Forbidden Submatrices

Ruiyuan (Ronnie) Chen

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This summer I worked with Dr. Richard Anstee on the problem of forbidden submatrices. We are given a $k \times \ell$ matrix F whose entries are 0 or 1 (henceforth a $(0,1)$-*matrix*), and asked what is the maximum number of unique columns a $(0,1)$-matrix A with m rows may have, without containing a copy of F as a submatrix. Our research was guided by the following conjecture:

Conjecture 1 (Anstee, Füredi [2]; Frankl, Füredi, Pach [3])**.** *Let F be a given $k \times \ell$ $(0,1)$-matrix. Then there exists a constant c_F such that for any $m \times n$ $(0,1)$-matrix A with no repeated columns and no submatrix F,*

$$n \leq c_F m^k.$$

The problem of forbidden submatrices belongs to extremal set theory, and is related to some other problems in that field. For example, the problem of forbidden *configurations* asks for bounds on matrices A that do not contain any row or column permutation of a given matrix F. In proving results about forbidden submatrices, it is sometimes useful to refer to known results about other extremal problems, that can be adapted to the forbidden submatrices setting.

1 Upper bounds by amortized analysis

One half of the problem consists of providing upper bounds on the number of columns that any A may have without containing F as a submatrix. Our work this summer yielded the following two results.

Theorem 1. *Let F be the $2 \times \ell$ matrix*

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \end{bmatrix}.$$

Then for any $m \times n$ $(0,1)$-matrix A with no repeated columns and no submatrix F,

$$n \leq (\ell - 1)\binom{m}{2} + m + 1.$$

Theorem 2. *Let F be the $2 \times \ell$ matrix*

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 1 & \cdots \end{bmatrix}.$$

Then for any $m \times n$ $(0,1)$-matrix A with no repeated columns and no submatrix F,

$$n \leq 6(\ell - 1)\binom{m}{2} + m + 2.$$

Both theorems verify Conjecture 1 for their respective families of F. We borrowed ideas from computer science for the proofs of these theorems. Specifically, we imagine an algorithm that marches through the columns of A from left to right, greedily looking for a copy of F on every pair of rows. Using amortized analysis, we prove that on average, most columns of A must contribute a column to the copy of F we are building up on at least one pair of rows of A.

Amortized analysis was previously used by Dr. Anstee to prove the following result about single-row F:

Theorem 3 (Anstee [1]). *Let F be the $1 \times \ell$ matrix*

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \cdots \end{bmatrix}.$$

For any $m \times n$ $(0,1)$-matrix A with no repeated columns and no submatrix F,

$$n \leq (\ell - 1)m + 1.$$

Theorems 1 and 2 extend this result and further demonstrate the utility of amortized analysis in the forbidden submatrices setting.

2 Lower bounds by constructions

The other half of the problem consists of providing lower bounds on the maximum number of columns A may have without containing F as a submatrix. We do this by providing rules for constructing matrices A with m rows, for any given m, that have no submatrix F. We obtained two results that together give good constructions for a large family of F.

Theorem 4. *Let F be a $k \times 2$ $(0,1)$-matrix with two nonidentical columns. Then there exists an $m \times n$ $(0,1)$-matrix A with no repeated columns and no submatrix F, such that*

$$n \geq \binom{m}{k} + \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{1} + \binom{m}{0}.$$

Note that this implies that the upper bound in Conjecture 1 cannot be of a lower order (than m^k) for 2-column F with nonidentical columns. The following previously known result complements Theorem 4 by giving optimal constructions for 2-column F with identical columns.

Theorem 5 (Anstee, Füredi[2]). *Let F be a $k \times 2$ $(0,1)$-matrix with two identical columns both equal to ϕ. Let t be the minimum number of blocks in a partition of ϕ into blocks of 1 or 2 consecutive entries, where each block is one of $[0], [1], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Then a maximal $m \times n$ $(0,1)$-matrix A with no repeated columns and no submatrix F has*

$$n = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{1} + \binom{m}{0} + \binom{m-t}{k-t}.$$

We may piece together the constructions for 2-column F from Theorems 4 and 5 to obtain good constructions for a large class of F.

Theorem 6. *Let F be a $k \times \ell$ $(0,1)$-matrix with a nonconstant top or bottom row. Let $\phi_1, \phi_2, \ldots, \phi_\ell$ denote the columns of F. Given $(m - \ell + 2)$-row $(0,1)$ matrices $A_1, A_2, \ldots, A_{\ell-1}$, each A_i with no repeated columns and no submatrix $[\phi_i \phi_{i+1}]$, we may construct a m-row $(0,1)$ matrix A with no repeated columns and no submatrix F by concatenating the A_i's side-by-side and then appending $\ell - 2$ rows above or below.*

References

[1] R. P. Anstee. On a conjecture concerning forbidden submatrices. *J. Combin. Math. Combin. Comput.* 32 (2000) 185-192.

[2] R. P. Anstee and Z. Füredi. Forbidden submatrices. *Discrete Math.* 62 (1986) 225-243.

[3] P. Frankl, Z. Füredi, and J. Pach. Bounding one-way differences. *Graphs Combin.* 3 (1987) 341-347.