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Egyptian Fraction Representations of One

Karlming Chen

Over the summer, I have worked under the supervision of Professor Greg Martin and Professor Mike Bennett on two problems regarding Egyptian fractions. An Egyptian fraction is a representation of a rational number r in the form

$$r = \sum_{i=1}^{n} \frac{1}{x_i}, \quad x_1 < x_2 < \ldots < x_n,$$

where all the x_i are positive integers. The first problem concerns specifically Egyptian fraction representations of 1 where $x_i \nmid x_j$ when $i \neq j$. Such representations exist, and the shortest known representation, found by Nechemia Burshtein, contains 52 terms [1]. I have focused on a special case of such representations, namely those where each x_i is of the form pq for primes p and q, which is incidentally the form of Burshtein's solution. The two problems that arise from this are whether a solution with fewer terms exists and whether a solution that contains a smaller largest prime exists. To this extent, some computer algorithms were devised for the purpose of finding representations using only the first n primes; the running time is estimated to be on or above the order of months to check for solutions when $n \geq 15$.

The second problem is concerned with the set $L_2(1)$, defined as:

$$L_2(1) = \{x \in \mathbf{Z}, x > 1 : \nexists x_1, \ldots, x_t \in \mathbf{Z}, x_1 > \ldots > x_t \geq 1 \text{ where } \sum_{i=1}^{t} 1/x_i = 1, x_2 = x\}.$$

This is the set of positive integers (greater than 1) that cannot be the second largest integer in an Egyptian fraction representation of 1. It was established in a paper by Greg Martin that $L_2(1)$ is finite [2], and it remains to ascertain what the set is. It would be possible to quantitatively determine the theoretical upper bound of $L_2(1)$, though this was not done. Progress was made, however, on explicitly determining the constants in the following subsidary result in [2]:

Take $r \in \mathbf{Q}$, $r > 0$. There exists a positive constant $\delta(r)$ where for every real $x > f(r)$ for some function f, all sets $\mathcal{E}$ of positive integers not exceeding x where $\sum_{n\in\mathcal{E}} 1/n = r$ satisfy:

$$|\mathcal{E}| \leq (1 - e^{-r})x - \delta(r)\frac{x \log\log x}{\log x}.$$

References:

[1] N. Burshtein, An improved solution of $\sum_{i=1}^{k} 1/x_i = 1$ in distinct integers when $x_i \nmid x_j$ for $i \neq j$, *NNTDM* 16 (2010), 2, 1-4

[2] G. Martin, Denser Egyptian fractions, *Acta Arithmetica* 95 (2000), 3, 231-259