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Perturbative and numerical solutions to the hydraulic fracture stress-jump problem

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This summer, I had the pleasure of conducting research with Prof. Anthony Peirce. We studied a problem in hydraulic fractures, whose origin lies in the application of the theory to mining conditions.

Hydraulic fractures are a class of brittle fractures that propagate in pre-stressed solid media due to the injection of a viscous fluid. In the oil and gas industry, hydraulic fractures are deliberately created in reservoirs to enhance the recovery of hydrocarbons by the creation of permeable pathways. The application of HF in geotechnical engineering is growing; for example, in the mining industry they have recently been used to weaken the rock and enhance the so-called block-caving process.

In a single axis of propagation, we define $w(x,t)$ to be the fracture width, $p(x,t)$ to be the corresponding pressure, and $l(t)$ to be the fracture half-length at time t. Given a stress jump $\sigma(x)$ and a fluid injection rate Q_0 at $x=0$, the problem can be formulated using a system of patrial differential equations

$$p(x,t)-\sigma(x) = -\frac{E'}{4\pi}\int_{-l(t)}^{l(t)} \frac{w(s,t)}{(s-x)^2}\,ds \tag{1}$$

$$\frac{\partial w(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\frac{w(x,t)}{\mu'}\frac{\partial p(x,t)}{\partial x}\right] + Q_0\delta(x) \tag{2}$$

further supplemented by fluid volume conservation

$$\int_{-l(t)}^{l(t)} w(x,t)dx = Q_0 t\ ,$$

and subject to boundary conditions

$$w(\pm l(t),t)=0, \qquad \frac{w(\pm l(t),t)^3}{\mu'}\frac{\partial p(\pm l(t),t)}{\partial x} = 0\ .$$

In past literature, the stress-jump problem $\sigma(x)=\sigma_0 \mathrm{H}\left(|x|-L\right)$was solved numerically; however, past algorithms have been somewhat slow and inaccurate. In past papers by Prof. Peirce, he developed the Hermite Culic Collocation Scheme (KGD) to solve the non-stress $\sigma(x)\equiv 0$ problem numerically. In my research, I first modified his algorithm to accomodate such a stress-jump. I did this exploiting the linearity of (1), and then analytically inverting the equation

$$-\sigma(x) = -\frac{E'}{4\pi}\int_{-l(t)}^{l(t)} \frac{\Delta w(s,t)}{(s-x)^2}\,ds\ .$$

This allows writing 1 in terms of the pressure as

$$p(x,t) = -\frac{E'}{4\pi} \int\limits_{-l(t)}^{l(t)} \frac{[w(s,t) - \Delta w(s,t)]}{(s-x)^2}\, ds \ ,$$

which allows to progress the pressure in time, and therefore to numerically solve the stress-jump problem. This modified algorithm (MKGD) turns out to be very fast and accurate.

In the case of large toughness — which is quantified in terms of the parameter K' becoming large — we have witnessed that the pressure approaches function that is constant over space. Hence, I continued by researching the model with perturbation theory. After quite some rescaling of the system in terms of dimensionless width $\Omega(\xi,\tau)$, pressure $\Pi(\xi,\tau)$ and length $\gamma(\tau)$, I expanded

$$\begin{aligned} \Pi(\xi,\tau) &= \Pi_0(\tau) + \Pi_1(\xi,\tau) + \dots \ , \\ \Omega(\xi,\tau) &= \Omega_0(\xi,\tau) + \Omega_1(\xi,\tau) + \dots \ , \\ \gamma(\tau) &= \gamma_0(\tau) + \gamma_1(\tau) + \dots \ . \end{aligned}$$

Deriving the corresponding equations for zero-th order of perturbation, I have been able to determine $\gamma_0(\tau)$ as the solution of

$$g_v \tau = \frac{g_k \pi}{2\sqrt{2}} \gamma_0^{\frac{3}{2}} + \frac{4}{g_e}\sqrt{\gamma_0^2 - 1}\ ,$$

for some constants g_v, g_k, g_e, and compute analytically that

$$\begin{aligned} \Omega_0(\xi,\tau) &= g_k \sqrt{\frac{\gamma_0}{2}}\sqrt{1-\xi^2} + \frac{4}{\pi g_e}\left\{ -\gamma_0 \xi \ln\left| \frac{\sqrt{1-\xi^2} + \xi\sqrt{\gamma_0^2 - 1}}{\sqrt{1-\xi^2} - \xi\sqrt{\gamma_0^2-1}} \right| + \ln\left| \frac{\gamma_0\sqrt{1-\xi^2} + \sqrt{\gamma_0^2-1}}{\gamma_0\sqrt{1-\xi^2} - \sqrt{\gamma_0^2-1}} \right| \right\} \ , \\ \Pi_0(\tau) &= \frac{g_k}{4\sqrt{2\gamma_0}} + \frac{1+\varepsilon}{g_e} - \frac{2}{\pi g_e} \sin^{-1}\left(\frac{1}{\gamma_0}\right) \ . \end{aligned}$$

Furthermore, I have been able to derive that the first-order solutions are given by

$$\begin{aligned} \Omega_1(\xi,\tau) &= \frac{\gamma_1}{\gamma_0}\Omega_0(\xi,\tau) + \frac{4\gamma_0}{\pi} \int\limits_0^1 \Pi_1(s,\tau) \ln\left| \frac{\sqrt{1-\xi^2} + \sqrt{1-s^2}}{\sqrt{1-\xi^2} - \sqrt{1-s^2}} \right| ds \ , \\ \Pi_1(\xi,\tau) &= \Pi_1(0,\tau) + g_m \gamma_0 \int\limits_0^{\xi} \frac{1}{\Omega(s,\tau)^3} \left\{ -\frac{g_v}{2} + \left(\frac{g_v}{\pi} - g_k \sqrt{\frac{\gamma_0}{2}}\dot{\gamma_0} \right) \xi\sqrt{1-\xi^2} + \frac{g_v}{\pi}\sin^{-1}(\xi) \right\} ds \end{aligned}$$

where the space-constants $\Pi_1(0,\tau), \gamma_1(t)$ are determined from the volume conservation and limit constraints.

These solutions approximate extremely well and are almost indistinguishable from the non-perturbative solutions $\Pi(\xi,\tau), \Omega(\xi,\tau)$ produced by the MKGD algorithm. The significance of these perturbative results lies in the fact that evaluating them is almost trivial, and thereby significantly less computationally-demanding than proceeding through any algorithms.