NSERC USRA Report - 2010 Summer

# Multiplication In Hecke Algebras


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This summer I worked under the direction of Dr. Masoud Kamgarpour. For most of the summer I spent time learning the following topics: linear algebra, p-adic number theory, Galois theory, and commutative algebra.

For the first three weeks I worked on a list of 30 problems in abstract linear algebra [3]. Over this period of time I managed to solve 13 of the 30 problems from the list. In the process of solving the problems I have learned some representation theory, commutative algebra and linear algebra. This is a sample problem that took me days to solve: Describe all the conjugacy classes of $GL_n(\mathbb{F}_q)$. There are four cases to consider and each depends on the eigenvalues of the matrix. The result is summarized in Table 1.

| Class Type (Eigenvalue Type) | Number of Conjugacy Class | Number of Elements from each Class |
|---|---|---|
| 1.) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ | $q-1$ | $1$ |
| 2.) $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ | $\frac{(q-1)(q-2)}{2}$ | $q(q+1)$ |
| 3.) $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ | $q-1$ | $(q+1)(q-1)$ |
| 4.) Quadratic extension case | $\frac{q(q-1)}{2}$ | $q(q-1)$ |

Table 1: Classification of the Conjugacy classes of $GL_n(\mathbb{F}_q)$.

For the class of type 4 on the table, the eigenvalues of the matrix lies strictly in the quadratic extension $\mathbb{F}_q^2$ of $\mathbb{F}_q$. Let $A_1, A_2, ..., A_n$ be representatives of distinct conjugacy classes. We know by the table that exactly $q-1$ of the $A_i$'s belongs to the Class of type 1, $\frac{(q-1)(q-2)}{2}$ belong to the class of type 2, and vice versa. Hence, by the table $n = q^2 - 1$ is the total number of classes. If we let $C_{GL_n(\mathbb{F}_q)}(A_i)$ be the conjugacy class containing $A_i$ then by the class formula we have that

$$\sum_{k=1}^{n} |G : C_{GL_n(\mathbb{F}_q)}(A_i)| = (q-1) + \frac{(q-2)(q-1)q(q+1)}{2} + (q-1)^2(q+1) + q(q-1) = (q^2-q)(q^2-1)$$

which is the number of elements in $GL_n(\mathbb{F}_q)$.

I then spent another three weeks learning about p-adic numbers. The main topics that I studied were completions of fields and algebraic closures. This included the construction of $\mathbb{Q}_p$ from $\mathbb{Q}$, the algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, and the completion $\Omega_p$ of $\overline{\mathbb{Q}_p}$. The field $\overline{\mathbb{Q}_p}$ itself is not complete because there exist a Cauchy Sequence that does not converge in $\overline{\mathbb{Q}_p}$. For example, $a_i = \sum_{k=0}^{i} b_k p^{n_k}$ where $b_i's$ are primitive $(p^{2^i} - 1)$th roots of 1 in $\overline{\mathbb{Q}_p}$ and $n_i$ is an increasing sequence of integers dependent on $a_i$ [2(Ch 3. Thm 12)]. The field $\Omega_p$ turns out to be algebraically closed and the proof can be found in [2 (Ch 3 Thm 13)].

Following that I studied Galois theory. I learned why there is no general solution in radicals to a polynomial equation of degree 5 or higher with coefficients in $\mathbb{C}$. The general polynomial of degree n means the polynomial

$$(x - x_1)(x - x_2)\cdots(x - x_n)$$

with roots $x_1, x_2, \cdots, x_n$ that are indeterminates. By definition, an algebraic element $\alpha$ over a field $F$ can be solved by radicals if $\alpha$ is an element of a field $K$ obtained by a chain of simple radical extensions

$$F = K_0 \subset K_1 \subset \cdots \subset K_t = K$$

where $K_{i+1} = K_i(\sqrt[n_i]{a_i})$ for some $a_i \in K_i$, $i = 0, 1 \cdots t-1$ [1 (Sec 14.7)]. From the following theorem,

**Theorem 1** *The polynomial $f(x) \in F[x]$ where $F$ is a field of characteristic 0 can be solved by radicals if and only if its Galois group is solvable* [1 (Sec 14.7)].

and the fact that the Galois group of the general polynomial of degree $n$ is isomorphic to $S_n$ we can conclude that no general solution in radicals for a polynomial equation of degree 5 or higher exists because $S_n$ is not a solvable group for $n \geq 5$.

For the remainder of the summer I studied commutative algebra and algebraic geometry. I focused on studying localization. The localization of a ring $R$ with respect to a multiplicatively closed subset $S$ of $R$ is denoted $S^{-1}R$ and is exact. For instance, if $0 \to L \to M \to N \to 0$ is an exact sequence of $R$-modules, then the induced sequence $0 \to S^{-1}L \to S^{-1}M \to S^{-1}N \to 0$ of $S^{-1}R$-modules is exact [4 (Ch 3 Sec 1 & 8)].

Overall, my summer has been a great learning experience. My mentor and I have decided to continue working together in the near year.

# References


[1] Dummit S. David., *Abstract Algebra*. 3rd Edition. John Wiley and Sons, Inc, 2004.

[2] Koblitz Neal., *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. 2nd Edition. Springer-Verlag New York Inc, 1948.

[3] *Problems in Linear Algebra and Representation Theory*. URL: http://www.masoudkamgapour.com/algebraproblems.pdf.

[4] Serge Lang., *Algebra*. 3rd Edition. Springer Science+Business Media LLC, 2002.