Forbidden Configurations and Repeated Induction

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Over this summer, I had the great privilege of working with Dr. Richard Anstee and Ph.D. student Miguel Raggi in the problem area of forbidden configurations, a topic in extremal set theory. The induction arguments employed in our research were similar to those used by Steven Karp, Dr. Anstee's 2008 USRA student [2].

It is convenient to use the language of matrix theory and sets for forbidden configurations. Let $[m] = \{1, 2, \ldots, m\}$. We define a *simple* matrix as a (0,1)-matrix with no repeated columns. An $m \times n$ simple matrix A can be thought of a family $\mathcal{A}$ of n subsets $S_1, S_2, \ldots, S_n$ of $[m]$ where $i \in S_j$ if and only if the i, j entry of A is 1. For example, if $m = 3$ and $\mathcal{A} = \big\{\emptyset, \{2\}, \{3\}, \{1,3\}, \{1,2,3\}\big\}$, then

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Assume we are given a $k \times \ell$ (0,1)-matrix F. We say that a matrix A has F as a *configuration* if a $k \times \ell$ submatrix of A is a row and column permutation of F. For example, for

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

we see that A as written above has F as a configuration, since the submatrix on the 2nd and 3rd rows, 4th and 5th columns, is a row permutation of F.

We define $\text{forb}(m, F)$ as the largest number of columns that a simple m-rowed matrix A can have subject to the condition that A contains no configuration F. Thus, any $m \times (\text{forb}(m, F) + 1)$ simple matrix contains F as a configuration. If A satisfies this property, we say F is a *forbidden configuration* in A.

For ease of communication, an assortment of special notation is used. Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows. Let $[A|B]$ denote the concatenation of two matrices A, B on the same number of rows. For an integer $q > 0$, we define $q \cdot A$ to be the concatenation of q copies of A, so that, namely, $2 \cdot A = [A|A]$. Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column of k 1's on top of ℓ 0's. Given an $m_1 \times n_1$ matrix A and an $m_2 \times n_2$ matrix B, let $A \times B$ be the $(m_1 + m_2) \times n_1 n_2$ matrix whose columns are obtained from placing every column of A on top of every column of B. We assume precedence of $\cdot$ over $\times$ and precedence of $\times$ over $|$ so that $[2 \cdot A|B \times C]$ is $[A|A|(B \times C)]$.

The following are fundamental results, the applicability of which our research has sought to extend.

Theorem 1 *Sauer[4], Perles and Shelah[5], Vapnik and Chervonenkis[6]. Assume k is given. Then*

$$\textit{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Theorem 2 *Gronau[3].* $forb(m, 2\cdot K_k) = forb(m, K_{k+1})$.

We have been able to augment these theorems by proving that these bounds hold for larger versions of these matrices. Some results of our research are below. Many of the new results come from repeated use of induction arguments that have been common in previous forbidden configurations research.

Theorem 3 *Let $k \geq 4$ be a given integer. Let α be a $k \times 1$ (0,1)-column consisting of at least two 1's and at least two 0's. For $m \geq k+1$,*

$$forb(m, [K_k|\alpha]) = forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

In fact, we can add more than one column.

Theorem 4 *Let p, q, k be given with $p, q \geq 2$ and $k > p+q$. Assume $\binom{p+q}{p} \geq k+2$. Then for $m \geq k+1$, $forb(m, [K_k|\mathbf{1}_p\mathbf{0}_q \times K_{k-p-q}]) = forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$. Moreover, for $4 \leq k \leq 14$ and $m \geq k+1$, $forb(m, [K_k|\mathbf{1}_2\mathbf{0}_2 \times K_{k-4}]) = forb(m, K_k)$.*

We expect that the result holds for more choices on p, q and our proof for the case $p = q = 2$ indicates some of the arguments that may help to show this.

When considering the matrix $2 \cdot K_k$, the use of Theorem 2 makes arguments for induction base cases simpler, allowing an easier proof for a larger matrix extension. Let

$$F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Theorem 5 *For $m \geq k+2$, $forb(m, [2\cdot K_k|F_1 \times K_{k-3}]) = forb(m, 2\cdot K_k)$.*

We also considered the problem of constructing a matrix such that every k-set of rows has a k-rowed configuration F, which is in some ways a polar opposite problem. Specifically, we define $\text{req}(m, F)$, for a k-rowed (0,1)-matrix F, as the minimum number of columns that an m-rowed simple matrix A can have ($m \geq k$) such that A contains F as a configuration on every k-set of rows. The following result can be obtained via a construction.

Theorem 6 *Let k be given.*

$$req(m, K_k) = O\left((\log m)^{k-1}\right).$$

Dr. Anstee, Miguel Raggi, and I have drafted a paper that includes these results. As it stands now, it seems that we have far from exhausted the potential of these methods and that much more could be proved. In particular, there are likely more examples of matrices G and H such that $\text{forb}(m, [K_k|G]) = \text{forb}(m, K_k)$ and $\text{forb}(m, [2\cdot K_k|H]) = \text{forb}(m, 2\cdot K_k)$. For more information on forbidden configurations, consult Dr. Anstee's survey paper [1].

References

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