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Spectra of Sol-Manifolds

Curt Da Silva

Working with Kevin Luk, under the supervision of Mahta Khosravi and Malabika Pramanik, we initially delved into the basics of differential geometry, culminating in the formulation of the Laplace-Beltrami operator, denoted Δ, for general Riemannian manifolds (which reduces to the standard Laplacian in Euclidean space). This operator is linear and Hermitian, which implies that its spectrum is discrete and diverges to infinity. As the Laplace-Beltrami operator is intimiately connected to the Riemannian metric of the manifold, its spectrum is rich with geometric information about the manifold and is thus a ready candidate for study. An important object in the study of the spectrum of the Laplace-Beltrami operator on Riemannian manifolds is the spectral counting function, which is defined as:

$$N(t) = |\{\lambda \in \sigma(T) : \lambda \text{ is an eigenvalue of } T \text{ and } \lambda < t\}|$$

Our summer project dealt with obtaining an accurate approximation to this function and studying its behaviour for a specific class of manifolds called Sol-manifolds. These are constructed in the following manner: given a matrix $A \in SL(2,\mathbb{Z})$ with eigenvalues λ, λ^{-1} and $\lambda \neq \pm 1$, this matrix defines an action on the space $\tilde{M}^3 = T^2 \times \mathbb{R}$ by $(x, y, z) \to (A(x, y), z + 1)$ with x, y coordinates computed modulo 1. The Sol-manifold M is defined as the quotient $\tilde{M}^3/\mathbb{Z}$ by this action. Intuitively, one can regard this construction as attaching a torus to each point along the circle S^1. In choosing a point (x, y) on a particular torus, say T_z, travelling once around the circle results in (x, y) being sent to $A(x, y)$ mod 1 on the same torus. The Sol-manifold is then constructed by identifying (x, y) and $A(x, y)$ in this manner.

Weyl's Law dictates that for a compact manifold M, $N(t)$ behaves asymptotically like $\frac{4}{3}\pi t^{n/2}(2\pi)^{-n}vol(M)$ where $n = dim(M)$. Moreover, we have that $N(t) = \frac{4}{3}\pi t^{n/2}(2\pi)^{-n}vol(M) + R(t)$, where the remainder term, $R(t)$, is in $O(t^{(n-1)/2})$, due to a result from Hörmander. The primary objective of our study this summer was to numerically approximate the behaviour of $N(t)$ in order to obtain a better idea of the constant involved in the big O term in the remainder for Sol-manifolds.

The spectrum of the Laplace-Beltrami operator on Sol-manifolds results in a partial differential equation which is amenable to separation of variables, and thus gives rise to eigenfunctions of the following form: $\Psi_\gamma(u, v, z) = e^{2\pi i(\gamma,(u,v))}f(z)$ where f is a solution to the modified Mathieu equation

$$\left(-\frac{d^2}{dz^2} + |\nu(\gamma)| \cosh(2\ln(\lambda)(z + \alpha(z)))\right) f(z) = \Lambda f(z)$$

and $\nu(\gamma), \alpha(\gamma)$ are constants which depend on $\gamma \in \Gamma^*$, the dual lattice associated to the gluing map A. The eigenfunctions Ψ_γ however, are only defined on $\tilde{M}^3$, the covering space of the manifold, rather than on the manifold M itsef, since they are not invariant under the action generated by A. True eigenfunctions on the manifold are then constructed by averaging these functions over the whole dual lattice, namely

$$\Phi_\gamma(u, v, z) = \sum_{n\in\mathbb{Z}} \Psi_{A^{*n}}(u, v, z)$$

The spectrum of Δ thus consists of two parts, the trivial part $\epsilon_k = (2\pi)^2k^2$, corresponding to $\gamma = 0$, and the nontrivial part, corresponding to the modified Mathieu equation. Thus computing an approximation for the eigenvalue counting function on a Sol-manifold involves computing a large number of eigenvalues corresponding to modified Mathieu equations associated to a large number of dual lattice points γ. The problem is computed on the covering space $\tilde{M}^3$ but the scope of the problem can be reduced by the results of Bolsinov et al., which state that computing eigenvalues for dual lattice points which evaluate to the same value of a particular quadratic form (associated to A) can be reduced to computing eigenvalues for a particular representative γ. Moreover, the multiplicities of this γ can be computed explicitly (which depends on the nature of the map A) itself. Results by Coïsson et al. provide a computationally efficient means of computing a large number of eigenvalues for each modified Mathieu equation.