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Minimal Surfaces and Complex Submanifolds

Jonathan Blackman

This summer I studied the relation between minimal surfaces and complex submanifolds with Dr. Ailana Fraser. It is well known that complex submanifolds, which have complex tangent spaces, minimize volume absolutely due to Wirtinger's inequality. This implies that they are stable minimal surfaces, meaning the first variation of the volume vanishes and the second variation is non-negative under compactly supported normal deformations. The question of when stable minimal surfaces are holomorphic under some complex structure arises naturally.

A complex version of the stability inequality for minimal surfaces was derived, including curvature terms for the case of an underlying space which is not flat. A theorem of Micallef, which makes use of the complex stability inequality, states that any complete parabolic two-dimensional surface in four-dimensional Euclidean space is holomorphic with respect to some orthogonal complex structure. This result was studied in detail, as well as various conditions which imply the parabolic growth condition such as quadratic geodesic area growth or finite total curvature. A theorem of Morgan was also investigated, which states that two intersecting planes in four-dimensional Euclidean space minimize area if and only if they are both holomorphic under some complex structure. These two theorems suggest that collections of parabolic surfaces which minimize volume and are stable may be holomorphic under the same complex structure.

In addition to studying these results in detail, I did some more general background reading to strengthen my knowledge of differential geometry and minimal surfaces. I studied the Plateau problem, which is to show the existence of a minimal surface with a given boundary. It was solved independently by both Douglas and Rado in 1930, and Douglas earned the Fields medal for his solution. Osserman's book on minimal surfaces provided me with a more general knowledge of minimal surfaces, such as the Enneper-Weierstrass representation of minimal surfaces in three dimensional Euclidean space, the fact that the coordinate functions of a minimal surface are harmonic and Bernstein's theorem which states that the only minimal surface in three-dimensional Euclidean space which is a graph is a plane. I also studied some more technical global results in Colding and Minicozzi's book on minimal submanifolds.