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Greenberg-Lubotzky's theorem for sheaves on graphs

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Bounding the second largest eigenvalue of d-regular graphs has been of great interest in graph theory. One of the main known results regarding the second largest eigenvalue is Alon-Boppana's lower bound, which states that the second largest eigenvalue of a (n, d)-regular graph is bounded below by $2\sqrt{d-1} - \alpha(n)$, where $\alpha(n) \to 0$ as $n \to 0$. If G is a bouquet of loops of total degree d, then every d-regular graph is a lift of G. Incidentally, the universal cover of G (i.e. d-regular infinite tree T_d) has spectral radius $2\sqrt{d-1}$. Therefore, it seems that the largest new eigenvalue of lifts of a graph is somehow related to the spectral radius of its universal cover. This relationship is established in the following theorem.

Theorem 0.1 (Greenberg-Lubotzky) *Let G be a fixed graph and ρ denote the spectral radius of its universal cover. Given $\epsilon > 0$, there exists N such that every covering of G, with covering degree greater than N, has a new eigenvalue of absolute value at least $\rho - \epsilon$.*

For my research project, I investigated the validity of Greenberg-Lubotzky's theorem for sheaves on graphs. Sheaf on a graph is a concept borrowed from topology that generalizes the notion of graph. In algebraic graph theory, each vertex is viewed as a one dimensional vector space; whereas, in a sheaf, we assign a multi-dimensional vector space to each vertex and edge. Many concepts from graph theory (such as adjacency matrix, covering maps, universal cover and spectral radius) can easily be generalized to sheaves on graphs.

We showed that Greenberg-Lubotzky's theorem, in general, does not hold for sheaves. However, there are two situations for which we can generalize Greenberg-Lubotzky's theorem for sheaves.

Definition 0.2 *Let G be a graph. For every vertex $v \in V(G)$, its* local girth *is the length of the shortest non-backtracking closed walk containing v. The* local girth of graph G *is the maximum of local girths over all vertices v in G.*

The following theorem is one of our main results. This theorem shows that in the case of sheaves, even though Greenberg-Lubotzky's theorem might not necessarily hold for all lifts with sufficiently large covering degree, it holds for most such lifts.

Theorem 0.3 *Let $\mathcal{G}$ be a fixed sheaf and ρ denote the spectral radius of its universal cover. Given $\epsilon > 0$, there exist N and D such that every cover of $\mathcal{G}$, with covering degree greater than N and local girth greater than D, has a new eigenvalue with absolute value greater than $\rho - \epsilon$.*

We also showed that for certain sheaves, namely *nonnegative* sheaves [1], Greenberg-Lubotzky's theorem holds; that is,

Theorem 0.4 *Let $\mathcal{G}$ be a fixed nonnegative sheaf and ρ denote the spectral radius of its universal cover. Given ϵ, there exist N such that every cover of $\mathcal{G}$, with covering degree greater than N, has a new eigenvalue with absolute value greater than $\rho - \epsilon$.*

[1] We have a precise definition for these sheaves.