SUMMER 2010 NSERC USRA REPORT
OPTIMAL TRANSPORTATION

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Optimal transportation is studied in various fields with many important applications in economics and logistics. We studied the properties of optimal mappings and the way to construct such mappings for both discrete and continous mass densities. Studying the paper by L. V. Kantorovich marks the beginning of this project. We learned that for a translocation to be minimal, it is necessary and sufficient that there exists a potential associated with the translocation. The approach is via minimizing the cost functional, called the Monge-Kantorovich problem. The objective is to minimize the cost functional

$$J[T] = \int_X c(x, T(x))\rho(x)dx$$

where T(x) is the mapping that will transport the mass from the source density, $\rho(x)$ to the target density, $\bar{\rho}$, and $c(x, y)$ is the cost function (representing the cost of transporting a unit mass from x to y). In addition, the mass must be conserved, i.e. $T_{\#}\rho = \bar{\rho}$, meaning T must push (foward) the density ρ completely into $\bar{\rho}$. To have T as an optimal map from $\rho$to$\bar{\rho}$with respect to c(x,y), $T = \nabla\phi$, with ϕ convex is required; more precisely, $graphT \subset \partial\phi$ for ϕ convex, smooth almost everywhere, where $\partial\phi$ denotes the subdifferential of ϕ. And the Monge-Ampere Equation is derived:

$$det(D^2(\phi(x))) = \frac{\rho}{\bar{\rho}(\nabla\phi(x))}$$

In the second part of the project, we studied the 1-D case for continous source and target densities, with respect to $c(x, y) = |x - y|^2$. To determine this map, I wrote a program that generate the graph of the optimal map T.

Besides the continuous target density, we also considered the case for discrete density. In this case, we could determine the optimal map T. The general form for the convex function is $\phi(x) = sup_y\{-c(x, y) - \bar{\phi}(y)\}$. To determine T, we only need to determine what $\bar{\phi}$ should be.

For the 1-D case with n discrete target densities, we concluded that $\bar{\phi}$ has to be a decreasing function of x. In addition, the exact value of $\bar{\phi}$ can be calculated by the computer program written by me. And the 2D case with 2 discrete targets can also be calcuated.

However, to determine $\bar{\phi}$ for the 2D case with 3 or more discrete targets appears to be a much more difficult problem. The difficulty lies in the fact that the geometry of the intersections can be hard to predict. We devised some ways to capture the different geometries. However, we could not write a computer program generating the optimal maps in this more general case.