Error Analysis of Projection Schemes

Siyavash Towfighi (supervised by Brian Wetton)

August 2, 2009

1 Introduction

The Navier-Stokes equations can be used to model the motion of many types fluids in different geometries. Over the years numerical methods have been developed to approximate their solutions to simulate fluid behavior, saving the industry millions of dollars by decreasing the need to create physical models to perform experiments. One class of these numerical methods used for time-stepping are the projection schemes originally presented by Chorin in reference [1]. The main advantage of these methods is that the resulting algebraic system is decoupled with symmetric positive definite matrices which can be solved efficiently, reducing computation time. In this project, an error analysis of different projection methods is presented and tested with a finite difference discretization in space. Although gauge methods [2], regular backward Euler, Crank-Nicolson and a spectral spacial discretization [5] were also implemented, they are omitted from this report due to space limitations.

2 Problem of Interest

It can be shown that the non-linear term in the Navier-Stokes equations does not affect the error analysis of the projection schemes as mentioned in reference [2]. Hence it suffices to consider the time dependent Stokes problem. Further simplifications can be made by assuming periodic boundary conditions in the x-direction and focusing on one component of the Fourier expansion [3]. The differential equations of interest have the form,

$$\frac{\partial u}{\partial t} + i\alpha p = -\alpha^2 u + \frac{\partial^2 u}{\partial y^2} + f_1 \tag{1}$$

$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = -\alpha^2 v + \frac{\partial^2 v}{\partial y^2} + f_2 \tag{2}$$

$$i\alpha u + \frac{\partial v}{\partial y} = 0 \tag{3}$$

with homogeneous boundary conditions for both u and v for $y \in (0, 1)$.

In order to verify theoretical results by numerical experiments, the Marker and Cell grid [4] was used for the finite difference spatial discretization. The test problem was taken to be,

$$u(y,t) = \frac{e^{i(\beta y+\omega t)}}{\alpha} y(2i - 6iy - \beta y + 4iy^2 + 2\beta y^2 - \beta y^3) \tag{4}$$

$$v(y,t) = e^{i(\beta y+\omega t)} y^2(1 - 2y + y^2) \tag{5}$$

$$p(y,t) = e^{i(\beta y+\omega t)} \tag{6}$$

with $\alpha = \beta = 1$ and $\omega = 2\pi$. Furthermore, all rates of convergence were recorded using only the final time step at $t = 1$ in order to avoid interference from errors due to discrete compatibility [6].

3 Pressure Correction Methods

A general formulation of the pressure correction methods can be given as follows:

$$\begin{cases} \frac{1}{k}\left(\beta_q \underline{\tilde{U}}^{n+1} - \sum_{j=0}^{q-1} \beta_j \underline{U}^{n-j}\right) - \Delta \underline{\tilde{U}}^{n+1} + \nabla P^{\star,n+1} = \underline{f}\left(t^{n+1}\right) \\ \underline{\tilde{U}}^{n+1}|_\Gamma = 0 \end{cases} \tag{7}$$

$$\begin{cases} \frac{\beta_q}{k}\left(\underline{U}^{n+1} - \underline{\tilde{U}}^{n+1}\right) + \nabla\left(P^{n+1} - P^{\star,n+1}\right) = 0 \\ \nabla \cdot \underline{U}^{n+1} = 0 \\ \underline{U}^{n+1} \cdot \hat{n}|_\Gamma = 0 \end{cases} \tag{8}$$

where k is the time step, β_i's are the coefficients of the backward differentiation formula, q is the order of the backward differentiation formula and $P^{\star,n+1}$ is an r^{th} order extrapolation of P^{n+1}. For example,

$$P^{\star,n+1} = \begin{cases} 0 & \text{if } r = 0, \\ P^n & \text{if } r = 1, \\ 2P^n - P^{n-1} & \text{if } r = 2. \end{cases} \tag{9}$$

Reference [2] contains more information about these methods and their implementation. The following section presents the results of the error analysis of these schemes.

3.1 Error Analysis

It can be shown by direct substitution that for $q = 2$ and $r = 0$, at the boundary $y = 0$ the calculated values $\underline{U}$ and P can be expressed as,

$$\underline{U} = \underline{u} + k\underline{u}^{(1)} + ... \tag{10}$$

$$P = p + k^{\frac{1}{2}}\sqrt{\frac{2}{3}} p_y(x,0,t) e^{k^{-\frac{1}{2}}\sqrt{\frac{3}{2}}y} + kp^{(1)} + ... \tag{11}$$

where $\underline{u}^{(1)}$ and $p^{(1)}$ are smooth functions that satisfy the forced Stokes problem,

$$\underline{u}_t^{(1)} + \nabla p^{(1)} = \Delta \underline{u}^{(1)} + \frac{2}{3}\Delta \nabla p \tag{12}$$

$$\nabla \cdot \underline{u}^{(1)} = 0 \tag{13}$$

$$\underline{u}^{(1)}|_{y=0} = \left(-i\alpha \frac{2}{3} p(x,0,t), 0\right) \tag{14}$$

Similarly, for $q = 2, r = 1$,

$$\underline{U} = \underline{u} + k^2 \underline{u}^{(2)} + ... \tag{15}$$

$$P = p + k\sqrt{\frac{2i\omega}{3}} p_{y,t}(x,0,t) e^{-m(k)y} + k^2 p^{(2)} + ... \tag{16}$$

where $m(k) = \sqrt{\frac{3}{2k^2 i\omega} + \alpha^2}$ and $\underline{u}^{(2)}$ and $p^{(2)}$ satisfy the forced Stokes problem,

$$\underline{u}_t^{(2)} + \nabla p^{(2)} = \Delta \underline{u}^{(2)} + \frac{1}{3}\underline{u}_{ttt} + \frac{2}{3}\Delta \nabla p_t \tag{17}$$

$$\nabla \cdot \underline{u}^{(2)} = 0 \tag{18}$$

$$\underline{u}^{(2)}|_{y=0} = \left(-i\alpha \frac{2}{3} p_t(x,0,t), 0\right) \tag{19}$$

These predicted rates of convergence were numerically tested using the problem defined in section 2 and the results were consistent.

4 Velocity Correction Methods

The general formulation of the velocity correction methods can be given very similarly to the pressure correction methods as follows:

$$\begin{cases} \frac{1}{k}\left(\beta_q \underline{U}^{n+1} - \sum_{j=0}^{q-1} \beta_j \underline{\tilde{U}}^{n-j}\right) - \Delta \underline{\tilde{U}}^{\star,n+1} + \nabla P^{n+1} = \underline{f}\left(t^{n+1}\right) \\ \nabla \cdot \underline{U}^{n+1} = 0 \\ \underline{U}^{n+1} \cdot \hat{n}|_\Gamma = 0 \end{cases} \tag{20}$$

$$\begin{cases} \frac{\beta_q}{k}\left(\underline{\tilde{U}}^{n+1} - \underline{U}^{n+1}\right) + \Delta\left(\underline{\tilde{U}}^{n+1} - \underline{\tilde{U}}^{\star,n+1}\right) = 0 \\ \underline{\tilde{U}}^{n+1}|_\Gamma = 0 \end{cases} \tag{21}$$

The notation used is identical the one presented for the pressure correction methods in section 3 and further details are available in reference [2].

4.1 Error Analysis

As in the previous section, the results for $q = 2, r = 0$ and $q = 2, r = 1$ are presented. For the case $q = 2, r = 0$,

$$\underline{\tilde{U}} = \underline{u} + \left(k^{\frac{3}{2}} i\alpha \left(\frac{2}{3}\right)^{\frac{3}{2}}, -k\frac{2}{3}\right) v_{yy}(x,0,t) e^{-m(k)y} + k\underline{\tilde{u}}^{(1)} + ... \tag{22}$$

$$P = p + k^{\frac{1}{2}} \left(\frac{2}{3}\right)^{\frac{1}{2}} v_{yy}(x,0,t) e^{-m(k)y} + kp^{(1)} + ... \tag{23}$$

$$\underline{U} = \underline{u} + k\underline{u}^{(1)} + ... \tag{24}$$

where $m(k) = \sqrt{\frac{3}{2k} + \alpha^2}$ and $\underline{\tilde{u}}^{(1)}$ and $p^{(1)}$ satisfy the forced Stokes problem,

$$\underline{\tilde{u}}_t^{(1)} + \nabla p^{(1)} = \Delta \underline{\tilde{u}}^{(1)} \tag{25}$$

$$\nabla \cdot \underline{\tilde{u}}^{(1)} = 0 \tag{26}$$

$$\underline{\tilde{u}}^{(1)}(x,0,t) = \left(0, \frac{2}{3} v_{yy}(x,0,t)\right) \tag{27}$$

Similarly for $q = 2, r = 1$,

$$\underline{\tilde{U}} = \underline{u} + \left(-k^3 \alpha \left(\frac{2}{3}\right)^{\frac{3}{2}} \sqrt{-i\omega}, -k^2\frac{2}{3}\right) v_{yy,t}(x,0,t) e^{-m(k)y} + k^2\underline{\tilde{u}}^{(2)} + ... \tag{28}$$

$$P = p + k\sqrt{-\frac{2i}{3\omega}} v_{yy,t}(x,0,t) e^{-m(k)y} + k^2 p^{(2)} + ... \tag{29}$$

$$\underline{U} = \underline{u} + k^2\underline{u}^{(2)} + ... \tag{30}$$

where $m(k) = \sqrt{\frac{3}{2k^2 i\omega} + \alpha^2}$ and $\underline{\tilde{u}}^{(2)}$ and $p^{(2)}$ satisfy the forced Stokes problem,

$$\underline{\tilde{u}}_t^{(2)} + \nabla p^{(2)} = \Delta \underline{\tilde{u}}^{(2)} + \frac{1}{3}\underline{u}_{ttt} \tag{31}$$

$$\nabla \cdot \underline{\tilde{u}}^{(2)} = 0 \tag{32}$$

$$\underline{\tilde{u}}^{(2)}(x,0,t) = \left(0, \frac{2}{3} v_{yy,t}(x,0,t)\right) \tag{33}$$

5 Conclusion

Based on the error analysis, one could assume with reasonable certainty that the solution from the projection methods is not as accurate at the boundaries. Although not shown in this report, it is possible to achieve higher rates of convergence by increasing the value of r. However for $r \geq 2$, there may be stability problems [2]. Hence projection methods are reliable generally for rates of convergence upto 2. However, in order to achieve higher accuracy, there are other methods available such as the semi-implicit projection methods based on spectral deferred corrections. Reference [7] discusses this method in more detail. I did not implement this method for the project but if there are further questions about anything else, feel free to send me an email (siyavash@interchange.ubc.ca).

References

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