NSERC USRA Report- Random Sorting Networks

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This summer, I studied the properties of random sorting networks with Dr. Omer Angel. A sorting network of n elements is a way of transforming the ordered set $\{1, 2, ..., n\}$ to the reverse-ordered set $\{n, n-1, ..., 1\}$ through a series of adjacent transpositions of elements. For example, one 4 element sorting network is,

$$\{1,2,3,4\} \to \{2,1,3,4\} \to \{2,3,1,4\} \to \{2,3,4,1\} \to \{2,4,3,1\} \to \{4,2,3,1\} \to \{4,3,2,1\}$$

This may also be considered as a sequence of permutations of n elements from the identity permutation to the reverse, where each permutation differs from the preceeding by a single transposition of two adjacent numbers. By considering the position of a single element as a function of 'time', or progress through the network, one arrives at the trajectory of a particle. Furthermore, by introducing the appropriate scaling factors n and $\binom{n}{2}$ of distance and time, respectively, one gets the scaled trajectories. A random sorting network of size n is a sorting network chosen uniformly at random from all possible, valid, sorting networks with n elements.

The analysis uses two prior results[1]: the circular distribution of the probability density function and the stationarity of swaps. It has been found that, as the number of elements n becomes arbitrarily large, the distribution of the scaled position of the first swaps follows a distribution of the form

$$P(x) = \frac{8}{\pi n}\sqrt{x(1-x)}.$$

The stationarity of swaps guarantees that this distribution holds for every swap of the sorting network, not simply for the first one. This is true since, for every sorting network, another sorting network may be found such that the first swap of the new network is the second swap of the original.

For this project, a stochastic model was developed for the particle trajectories showing heuristically that the limiting trajectories of random sorting networks approach sine curves with a range of amplitudes and phases. This model was generated with the following assumptions: the swap locations are truly independent and they follow the above circular distribution. Furthermore, for the sake of the model, each particle's path will be considered to be independent of all other particle's movements. With these simplifications, and as $n \to \infty$, one arrives at the functional, dependent upon the position x and the total number of swaps y that a particle has made as:

$$\mathbb{P}(x,y) \sim c \exp -n \int_0^1 \left(\frac{4}{\pi}\beta - y'(t)\log\beta + \frac{x'(t)+y'(t)}{2}\log\left[x'(t)+y'(t)\right] \right.$$
$$\left. + \frac{y'(t)-x'(t)}{2}\log\left[y'(t)-x'(t)\right] \right) dt.$$

Using standard methods from the calculus of variations, this functional can be turned into a differential equation, the solution to which is an extremum of the functional. A solution to the corresponding equation is:

$$x(t) = \sin(\omega t + \delta).$$

This result yielded the expected sinusoidal behaviour, but interestingly, not with all the desired properties, namely, an arbitrary amplitude. This suggests that a more complicated model, which takes into account interactions between the particles, should be considered.

Work was also done in studying the bijection which exists between random sorting networks and staircase Young tableaux. In particular, possible configurations for entries along the diagonal of a staircase tableau were studied. A formula for the probability of a given shape in the Young tableaux occurring, or a particular configuration forming, was determined. It was found that for a given set S of distinct shapes λ within a given configuration, at scaled position y,

$$P \approx \prod_{\lambda \in S} \lambda \frac{2^{2|\lambda|}}{n^{|\lambda|}} |\lambda|! \sqrt{1-\bar{y}^2}^{|\lambda|} \prod_{\lambda_i \in \lambda} [\frac{1}{\binom{2\lambda_i}{\lambda_i} \Gamma(\frac{\lambda_i}{2}+1)}]^2 \epsilon(\lambda_i) [\prod_{l=1}^{i-\lambda_i} 1 - \frac{\lambda_i \lambda^{[i+\lambda_i-l+1]}}{\chi^2 - \frac{(\lambda_i + \lambda^{[i+\lambda_i-l+1]})^2}{4}}]$$

$$\prod_{\lambda_i, \lambda_j \in S} \prod_{i<j} \prod_{rows \lambda_i^k \in \lambda_i} \prod_{cols \lambda_j^l \in \lambda_j} [1 + \frac{2|\lambda_i^k||\lambda_j^l|}{\Delta[\Delta - (|\lambda_i^k| + |\lambda_j^l|)]}],$$

where Δ is approximately twice the difference between the elements. The next step would be to use this equation to determine the limiting shape and positional distribution of the sub-tableaux elements.

References

[1] Omer Angel, Alexander E. Holroyd, Dan Romik, and Balint Virag. Random sorting networks, 2006.