

NSERC USRA Research Report  
Oda's Strong Conjecture  
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I worked this summer with Dr.Kalle Karu of the UBC Math Department. Oda's strong conjecture has been a long standing open problem in algebraic geometry. The weak form of the conjecture is proven and can be used to prove the strong form.

**Conjecture 1.** *Every proper and equivariant birational map  $f : X_{\Delta} \rightarrow X_{\Delta'}$  between two nonsingular toric varieties can be factored into a sequence of blowups and blowdowns with smooth centres which are the closures of orbits.*

A sequence consisting of blowups and blowdowns in any order is called a *weak factorization*, and one with blowups immediately followed by blowdowns is a *strong factorization*.

This challenging problem can be reformualted into the language of nonsingular rational polyhedral cones. Here, a star subdivision and assembling corresponds to the abstract notion of a blowup and blowdown respectively. More generally, a fan is a collection of cones. The conjecture can then be reworded as:

**Conjecture 2.** *Let  $\Delta$  and  $\Delta'$  be two nonsingular fans in  $N_{\mathbb{Q}}$  with the same support. Then there is a sequence of smooth star subdivisions and inverse operations called smooth star assemblings starting from  $\Delta$  and ending with  $\Delta'$*

We will allow for one further simplification. Each star subdivision or assembling can be represented by a  $3 \times 3$  transvection  $E_{ij}$ , that is a matrix with 1's along the diagonal and  $ij^{th}$  entry and 0's everywhere else. Thus a sequence of blowdowns and blowups can be represented by a sequence of transvections; inverses followed by non-inverses. Finally, operations can be derived on how to move an inverse to the right of a non-inverse.

A sequence is resolvable if all inverses can be moved to the right of non-inverses in a finite amount of operations. Considering these transvections, the problem now becomes: Given a finite sequence of inverse and non-inverse transvections, is it possible to resolve the sequence in a finite amount of operations?

This idea was explored further this summer. The main area of issue was dealing with the fact that one operation has a choice. That is,

$$E_{ij}^{-1} E_{ik} \Rightarrow E_{ik} E_{ij}^{-1} \text{ or } E_{ki} E_{jk} R_{ijk} E_{kj}^{-1} E_{ji}^{-1} \quad (1)$$

Here  $R_{ijk}$  is a rotation matrix. This operation could potentially lead to an infinite process. For example, the operation might create two more points of choice, each creating their own and so on.

A directed sequence,  $D(i)$  is one containing transvections of the form  $E_{ij}, E_{ik}$ , that is those whose first index is the same. We could show, among other things, that all sequences of the form  $D^{-1}(i)D(j)$ , possibly with  $i = j$ , can be resolved.

**Example.**

$$\begin{aligned} & (E_{ij}^{-1})^m (E_{ik}^{-1})^n E_{kj}^p \\ &= (E_{ik}^{-1})^n (E_{ij}^{-1})^m E_{kj}^p \text{ since } E_{ij}^{-1} \text{ and } E_{ik}^{-1} \text{ commute} \\ &= (E_{ik}^{-1})^n E_{kj}^p (E_{ij}^{-1})^m \text{ since } E_{ij}^{-1} \text{ and } E_{kj} \text{ commute} \\ &= E_{kj}^p (E_{ij}^{-1})^{pn} (E_{ik}^{-1})^n (E_{ij}^{-1})^m \\ &= E_{kj}^p (E_{ij}^{-1})^{pn+m} (E_{ik}^{-1})^n \end{aligned}$$

Most cases are straightforward except for  $D^{-1}(i)D(i)$ . The main result of the summer was showing that this is resolvable.

**Theorem 3.** *All factorizations of  $(E_{ij}^{-1})^m E_{ik}^n$  are finite and of the form  $ST^{-1}$  where  $S$  and  $T$  are sequences of groups  $H(i,j,k)$  with*

$$H(i, j, k) = (E_{ij} E_{ik})^{m_1} E_{ji}^{n_1} \dots (E_{ij} E_{ik})^{m_k} E_{ji}^{n_k} E_{ij} E_{ki}$$

and  $m_1, n_1, \dots, m_k, n_k$  are nonnegative integers.

The remaining part is to use these important pieces to resolve an arbitrary sequence. Although it should be relatively easy, it is showing to be more difficult, as no simple pattern can be recognized.

Dr. Karu and I will be continuing to pursue a solution to the remaining portion and possible other methods, while at the same time working on producing a paper containing the results thus far.