Methods of Harmonic Analysis

Summer 2009, NSERC USRA Report   Hannah Cairns

I spent most of my research time this summer studying the basic methods of harmonic analysis. My goal was to write an explanation and elaboration of Fefferman's 1973 paper, *Pointwise Convergence of Fourier Series*. I was attempting to fill in the details and clarify things to the degree that the paper would be accessible at the level of a beginning graduate student, and to add enough background that the paper could be read more or less without reference to the literature.

The outcome of this summer research was that I now have more background in some special parts of real analysis like singular integral theory and interpolation theorems. The paper proved to be too difficult for me to deal with in the time I had. So in this report I'll talk about the background which I got from this experience.

Fefferman's paper [2] gives another proof of the Carleson-Hunt theorem.

Carleson-Hunt theorem. Let $f \in L^p[0, 2\pi]$ for $1 < p < \infty$. Then except on a set with Lebesgue measure zero, the partial sums f_n of the Fourier series, defined as follows,

$$f_n(x) = \frac{1}{2\pi} \sum_{j=-n}^{n} e^{ijx} \int_0^{2\pi} e^{-ijy} f(y) \, dy,$$

converge pointwise to a limit as $N \to \infty$.

This was proven first for $p = 2$ in 1966 by Lennart Carleson [1], and it was extended to all $1 < p < \infty$ by Richard Hunt [3]. The theorem was a long-standing open problem in the field of harmonic analysis at the time that it was solved.

When I began reading the paper, the first thing that I didn't understand was this. It is 'well-known' that the following integral,

$$\int_{-\infty}^{\infty} \frac{f(x-y)}{y} \, dy,$$

exists in the Cauchy principal value sense *almost everywhere* in x for any $f \in L^p(\mathbb{R})$ ($p < \infty$). At first glance that is really surprising. So I determined to learn more about this, and I began the summer by reading a work by Stein [6] on the L^p theory of singular integrals.

First, I had to look at Rudin [5] and Stein and Weiss [7] for the construction of the Fourier transform on the real line and its properties. From this and from working with Fefferman's paper, I got practice in dealing with the situation where I had information (for example, a bound on an operator) only on a dense subset of a Banach space.

Stein's book [6] in particular relies on L^p operator interpolation theorems, so I focused on those for some time. The Marcinkiewicz [6] [4] and Riesz-Thorin [7] [4] interpolation theorems were appealing despite the difficulty of their proofs. Terence Tao has remarked on his blog [8] that the bound from the Marcinkiewicz theorem can be 'amplified' by passing to larger and larger tensor powers of the function space, which avoids the use of complex analysis. It's nice to know that there is another way to do it.

The final part of the summer was spent going over Fefferman's paper and attempting to understand it on a rigorous level. The paper assumes a high level of mathematical maturity. In particular, it makes many "formal" manipulations, like moving a limit through an infinite sum and integral, where the justifications are neither explicit nor obvious to me; convincing myself took a lot of guessing and trial and error.

While I have no material results to show for my summer, I did receive a broad exposure to a field which I was completely unfamiliar with beforehand. I found the effort very enjoyable and worthwhile.

Thank you for the opportunity to participate in this program.

References

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