Double-Critical Graphs

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Abstract

A graph G is said to be double-critical if G is connected and the chromatic number of G decreases by 2 when any two adjacent vertices of G are removed. In 1966, Lovász conjectured that the only k-chromatic double-critical graph is K_k. The conjecture has been resolved for $k \leq 5$ but remains open for $k \geq 6$. In this paper, we prove various properties of double-critical graphs. In particular, we show that any k-chromatic double-critical non-complete graph has minimum degree at least $k+1$ and connectivity at least 6. For $k = 6$, we show that the minimum degree is at least $k + 2 = 8$, and the neighbours of a vertex of degree 8 must induce a subgraph that is isomorphic to one of two possible graphs. For $k = 7$, we show that the neighbours of a vertex of degree 8 must induce a $\overline{C_8}$. For $k = 8$, we show that the neighbours of a vertex of degree 9 must induce a $\overline{C_8} \vee K_1$ or a $\overline{C_9}$. We also show that any double-critical non-complete graph must contain at least 14 vertices.

1 Introduction

In this paper, all graphs are finite and undirected and contain no self-loop and no multiple edges. Let G be a graph. The set of vertices and the set of edges of G are denoted by $V(G)$ and $E(G)$, respectively. An edge of G is a set consisting of exactly 2 distinct vertices of G.

Let x and y be any two vertices of G. For convenience, we shall write xy for $\{x, y\}$, regardless of whether or not $\{x, y\}$ is an edge of G. The *open neighbourhood* of x in G, denoted $N_G(x)$, is the set of all vertices in G that are adjacent to x. The *closed neighbourhood* of x in G, denoted $N_G[x]$, is the set $\{x\} \cup N_G(x)$. We also define $T_G(x, y)$ by $T_G(x, y) = N_G(x) \cap N_G(y)$, which is the set of all vertices of G that are adjacent to both x and y. When it is clear from context what the underlying graph is, we shall omit the subscript and simply use $N(x), N[x]$, and $T(x, y)$.

Let A be any set of vertices in G. We shall use $G[A]$ to denote the subgraph of G induced by A, i.e., the subgraph of G consisting of the vertices in A and the edges of G where both endpoints are in A. A is said to be an *independent set* of G if no vertex in A is adjacent to another vertex in A, i.e., $G[A]$ contains no edges. A is said to be a *k-clique* if $|A| = k$ and every vertex in A is adjacent to every other vertex in A.

We shall use $\overline{G}$ to denote the complement of G, which is the graph with vertices $V(G)$, and for every $x, y \in V(\overline{G})$, we have $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. Let H be a graph whose vertex set $V(H)$ is disjoint from $V(G)$. We define the *join* of G and H, denoted $G \vee H$, to be the graph with vertices $V(G) \cup V(H)$ and edges $E(G) \cup E(H) \cup \{uv \mid u \in G \text{ and } v \in H\}$. We shall use $G - x$ to denote the subgraph $G[V(G) \setminus \{x\}]$, and we shall use $G - xy$ to denote the subgraph of G with vertices $V(G)$ and edges $E(G) \setminus \{xy\}$. We shall denote the complete graph, the cycle, and the path on n vertices by K_n, C_n, and P_n, respectively.

A *k-colouring* of G is a function $c : V(G) \to \{1, \dots, k\}$ such that $c(x) \neq c(y)$ for all $xy \in E(G)$. G is said to be *k-colourable* if there exists a k-colouring of G, and the *chromatic number* of G, denoted $\chi(G)$, is the least integer k for which G is k-colourable. G is said to be *vertex-critical* if $\chi(G - v) = \chi(G) - 1$ for all $v \in V(G)$, and said to be *edge-critical* if $\chi(G - xy) = \chi(G) - 1$ for all $xy \in E(G)$. If G is both vertex-critical and edge-critical, we simply say that G is *critical*. An edge $xy \in E(G)$ is said to be *double-critical* if $\chi(G - x - y) = \chi(G) - 2$.

G is said to be *double-critical* if G is connected and every edge of G is double-critical. It is clear that every complete graph is double-critical, but no other double-critical graphs have been found. In 1966, Lovász conjectured that the only k-chromatic double-critical graph is K_k. This is easily proven for $k \leq 4$, and in 1986, Stiebitz proved the conjecture for $k = 5$. The conjecture is still open for $k \geq 6$.

In this paper, we prove various properties of double-critical graphs. We show that every k-chromatic double-critical non-complete graph has minimum degree at least $k + 1$. For $k = 6$, we improve this lower bound to $k + 2 = 8$, and we investigate the subgraph induced by $N(v)$ for a vertex v with degree 8. We show that there are only two possible subgraphs that $N(v)$ can induce, up to isomorphism. For $k = 7$, we also investigate the subgraph induced by $N(v)$ for a vertex v with degree 8. In this case, there is only one possible subgraph that $N(v)$ can induce, namely $\overline{C_8}$. For $k = 8$, we show that for a vertex v with degree 9, there are only two possible subgraphs that $N(v)$ can induce (up to isomorphism), namely $\overline{C_8} \vee K_1$ and $\overline{C_9}$.

We also show that any double-critical non-complete graph has connectivity at least 6 and contains at least 14 vertices.