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I worked this summer with Dr. Richard Anstee of the UBC Math Department. I studied Forbidden Configurations, which is a topic in combinatorics. Generally speaking, in Forbidden Configurations we try to find the greatest number of subsets we can choose from $\{1, 2, \ldots, n\}$, subject to a certain restriction. Specifically, for any collection of subsets of $\{1, 2, \ldots, n\}$, we consider its incidence matrix A, which is an n-rowed $\{0, 1\}$ matrix with no repeated columns (the order of the columns is unimportant). For example, for $n = 3$ the collection $\{\emptyset, \{1\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3, 4\}\}$ is represented by the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let F be any $\{0, 1\}$ matrix (possibly with repeated columns), which we call our *forbidden configuration*. We say that an incidence matrix A *has the configuration* F if some submatrix of A is a row and column permutation of F. For example, consider the forbidden configurations

$$F_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrix A does not have the configuration F_1, but A does have the configuration F_2 (for example, taking rows 3 and 4, and columns 1, 3 and 5).

Given a forbidden configuration F and a positive integer n, we want to find the greatest number of subsets we can choose from $\{1, 2, \ldots, n\}$ so that its incidence matrix does not have the configuration F. We call this number $\text{forb}(n, F)$. I worked on finding exact answers for 2-columned forbidden configurations F, as well as for 3-rowed forbidden configurations. We were able to determine $\text{forb}(n, F)$ exactly for most 2-columned forbidden configurations which have one column with at most one 0. We also found some new proofs when F is a 3×3 matrix, and we were able to generalize these arguments to obtain exact results for a family of larger matrices. Our arguments made great use of induction. One of our challenges was developing suitable induction hypotheses, in which we often made use of assumptions about which $n \times \text{forb}(n, F)$ matrices do not have the configuration F (that is, the *extremal matrices*). Another challenge was proving our base cases, for which we usually employed counting arguments which were tailor-made to the particular forbidden configuration F. Interestingly, sometimes we came up with an inductive argument but were not able to establish the corresponding base case.

Dr. Anstee and I are in the process of writing a paper which includes these results. While we have made great progress, we do not think by any means that we have exhausted the techniques we used, and predict that they can be used to solve more Forbidden Configurations problems. I also note that while I was especially fond of using induction, many other techniques have been used to prove results in this topic, including graph theory and linear algebra. More information about Forbidden Configurations is available in this survey paper by Dr. Anstee: `http://www.math.ubc.ca/~anstee/FCsurvey05.pdf`.