Summer Research Summary

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During my discussions with Prof.**Rolfsen** we studied mostly on algebraic properties of groups[1]. We encountered many interesting groups. To name a few, Artin and Coxeter groups, Thompson group, group of orientation preserving homeomorphisms of the real line or $[0,1]$, group of germs of orientation preserving homeomorphisms of real line fixing one specific point.

Let G be a weighted graph (to every edge of a graph there is an associated number, $m(e_{x,y})$ that can be in $\{2,3,4,\ldots,\infty\}$)

$$\text{Artin}(G) = \langle v \in V_G : vwv\ldots = wvw\ldots \forall v,w \in V_G \rangle$$

$$\text{Coxeter}(G) = \langle v \in V_G : vwv\ldots = wvw\ldots \forall v,w \in V_G, v^2 = 1 \forall v \in V_G \rangle$$

In the above presentations $vwv\ldots = wvw\ldots$ means that on eachside of the equality we have $m(e_{v,w})$ letters.

$$\text{homeo}_+(\mathbb{R})$$

$$\text{homeo}_+([0,1])$$

$$\text{germ}_0(\text{homeo}_+([0,1]))$$

My first task started with reading and checking the computations in Prof.Rolfsen's paper on local indicability of Artin groups. As a sub-task I was assigned to learn about Schreier[2]-Reidemeister subgroup presentation method. and Also see whether or not the fact about Artin group B_3 which was forwarded to me originally from Ivan Morin was correct or not [*Is the braid on top of this page the trivial braid ?*] . During these activities I started writing a graphic program which would produce a braid diagram from an encoded word element of the braid group

braid $= [1,1,1,1,1,-2,3,3]$

Meanwhile I was assigned to check whether or not $F_4 = \langle a,b,c,d : aba = bab, bcbc = cbcb, dcd = cdc, ac = ca, ad = da, bd = db \rangle$ one of the exceptional Artin groups is locally indicable or not [which a positive answer to it would imply that this group is right-orderable]. My first attempt to solve this problem failed due a very naive mistake [which I tried to use the fact that link groups are locally indicable , I was looking for a link with its fundamental group F_4 , if such a link exists it's simple to show that it should have exactly 2 components.]

simple question. If all the relations of the group are commutator relations, is it true that the group is locally indicable ? How about if we assume that it doesn't have torsion element ? or under any other simple restriction .

As I was busy thinking about the F_4 problem, I found out pure Artin groups [colored Artin groups] have very nice properties . All the nice properties came from Artin's genius approach to organize group elements into some special intuitive form, known as Artin's combing. Now the general property is if you remove bunch of hyperplane in $\mathbb{R}^{2n}$ and compute the first homotopy group of this space, it can be shown that it decomposes into semi-direct product of bunch of free groups .

fact. $F_1 \rtimes F_2 \rtimes \ldots \rtimes F_n$ is locally indicable [where F_i's are free groups] . Moreover if the semi-direct products acts nicely you can prove that the resulting group is bi-orderable.

Related to these game-theory-like playing in high dimensions are 2 intuitive papers **Configuration Spaces**[10] and **Braid Groups**[1] .

Next topic which we worked on was about space of ordering of a group. There are good papers of **Sikora**[2] , **Navas**[5] and **Tatarin**[32]. We found many bi-orderings of Thompson group [the one defined as homeomorphisms of $[0,1]$ with breaking points on dyadically rational points and with slope in powers of 2] .

Them we showed that $\text{homeo}_+([0,1])$ embeds into $\text{germ}_0(\text{homeo}_+([0,1]))$ preserving some specific ordering of them and also as a group injection.

One other thing we did was reading an article about twisted Alexander polynomial which contained the following unclear lemma [in a manner that they are some operations not fully explained] :

[**Shapiro's Lemma**] If M is a π-module and $\kappa \subset \pi$, then $H_*(X_\kappa, M) = H_*(X, M \otimes_R R[\pi/\kappa])$, where X_κ is the cover of X associated to the subgroup κ and $M \otimes_R R[\pi/\kappa]$ is a π-module via the diagonal action.

and here is the exact concise [more algebraic] form of the above lemma[3]

[**Second Shapiro Lemma**]. If $H \le G$ and M is an H-module, then

$$H_*(H,M) \cong H_*(G, \text{Ind}_H^G M)$$

$$H^*(H,M) \cong H^*(G, \text{Coind}_H^G M)$$

Also, if you ever happened to check whether a group is locally indicable or not, it is enough to check this for only all the groups generated by finite number of generators and relations of your main group. [If you think finitely presentable groups are much easier to handle .]

List of Problems

1. Does there exists a finitely generated ordered group with an isolated ordering such that the semi-group of positive elements of that ordering is not finitely generated semi-group ?

   $\langle x, y : x^i y x^{-i} = y^{\frac{1}{p^i}} \forall i \in \mathbb{Z} \rangle$

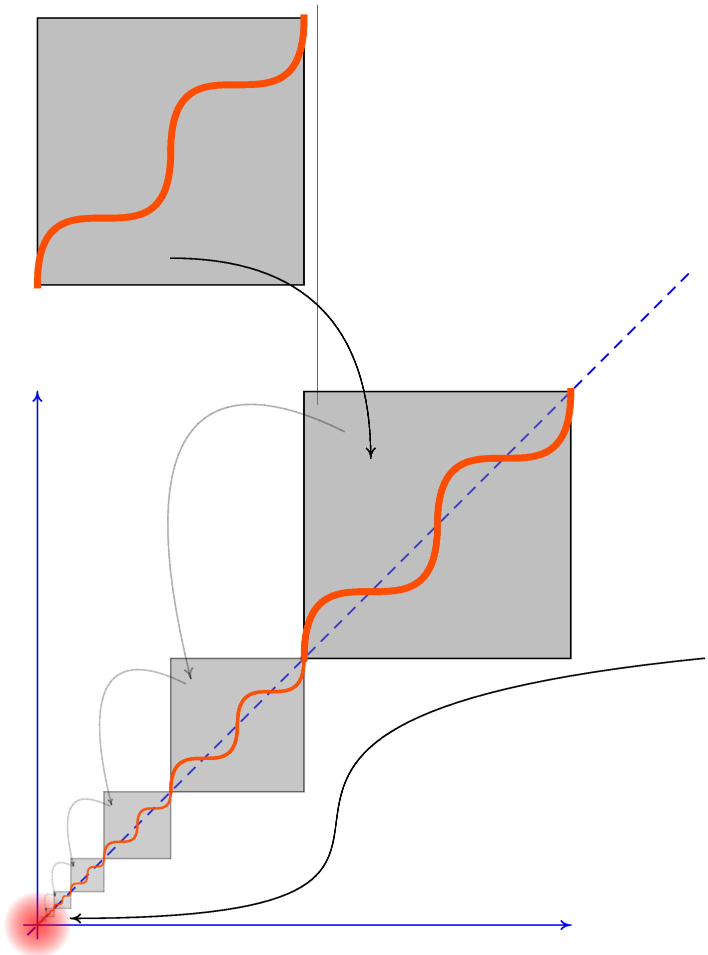
2. Right now no further problem comes to my mind or can be found in my notes but there are tons of interesting problems lurking here and there for .... .

*summer 2008:May-August

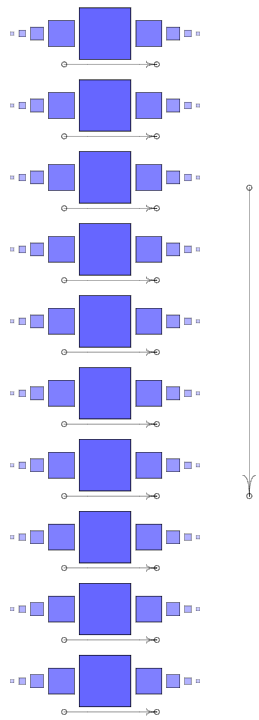
[1] orderability, local indicability , etc

[2] Otto Schreier was born in Vienna at 1901. His contemporaries were Hans Han, Kurt Reidemeister, Karl Menger, Wilhelm Wirtinger, Uryson, Alexandrov, Emil Artin ,etc. He was a talented mathematician contributed mostly to group theory and topology. Unfortunately, He died at a very young age of 28 in 1929. I don't have information on his cause of death.

[3] this lemma has never been published by Arnold Shapiro but was widely known to some group of people. Shapiro is among the mathematicians who died young , at the age of 41.



embedding $\text{homeo}_+([0,1])$ in $\text{germ}_0(\text{homeo}_+([0,1]))$



useful sources

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The blue diagram above represents graphically an action of a group on another group such that the whole systems makes a bigger group.