Positivity of the Chromatic Symmetric Function
NSERC USRA Report

Adrian She
Supervisors: Dr. Samantha Dahlberg and Dr. Stephanie van Willigenburg

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Let G be a finite, simple graph with vertex set V and edge set E. A **colouring** of G is a map $\kappa : V \to \mathbb{Z}$ satisfying $\kappa(u) \neq \kappa(v)$ if (u, v) is an edge in the graph. While graph colourings have numerous practical applications, one may also study them algebraically. The primary object studied in this USRA project was the **chromatic symmetric function** of a graph G. It was a generalization of the chromatic polynomial introduced by Richard Stanley in [2], and is defined as the formal power series

$$X_G = \sum_{\kappa : V \to \mathbb{Z}} x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}$$

where κ denotes a colouring of G. Figure 1 shows the some terms of the chromatic symmetric function of the path P_4.

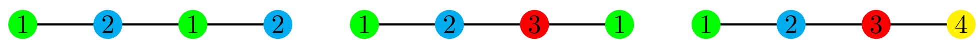


Figure 1: Some colourings of the path P_4, which show that $X_{P_4} = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2 x_3 x_4 + \cdots$.

It is conjectured that the chromatic symmetric function may distinguish non-isomorphic trees. Furthermore, if G is an n vertex graph, one may expand X_G into the sum $X_G = \sum_{\lambda \vdash n} c_\lambda u_\lambda$, where λ denotes a partition of n and $\{u_\lambda : \lambda \vdash n\}$ is some basis for the space of symmetric functions. As proved in [2], the coefficients c_λ, depending on the basis u_λ, reveal combinatorial information about the graph G, such as the number of partitions of its vertices into independent sets, its connected partitions, and the number of acylic orientations of its edges with a given number of sinks. We will often write $[u_\lambda]X_G$ to denote the coefficient c_λ when X_G is expanded into $X_G = \sum_{\lambda \vdash n} c_\lambda u_\lambda$.

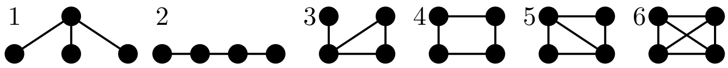
This USRA project was primarily focused on studying positivity of X_G in various bases due to connections with algebraic geometry and representation theory. Let $\{e_\lambda\}$ denote the set of **elementary symmetric functions** and $\{s_\lambda\}$ denote the set of **Schur functions**. The definitions of these bases can be found in various references, including Chapter 7 of [3]. We say a graph is ***e*-positive** if in the expansion $X_G = \sum_{\lambda \vdash n} c_\lambda e_\lambda$, each c_λ is non-negative. Similarly, we say that a graph is **Schur positive** if in the expansion $X_G = \sum_{\lambda \vdash n} c_\lambda s_\lambda$, each c_λ is non-negative.

To study e-positivity or Schur-positivity of graphs, we introduce a relation on the set of connected graphs on n-vertices, denoted by $\mathcal{C}_n$. If G and H are connected n-vertex graphs, then we say $G \geq_e H$ if the symmetric function $X_G - \frac{[e_n]X_G}{[e_n]X_H} X_H$ is e-positive. Similarly, we say that $G \geq_s H$ if the symmetric function $X_G - \frac{[s_{1^n}]X_G}{[s_{1^n}]X_H} X_H$ is Schur positive. Upon identifying graphs with the same chromatic symmetric function, $(\mathcal{C}_n, \leq_e)$ and $(\mathcal{C}_n, \leq_s)$ are partially ordered sets (posets). Figure 2 shows the posets for the case $n = 4$. We studied these posets in this USRA project, with the main results summarized as follows.

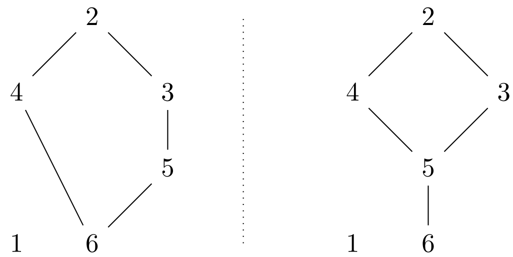
Firstly, the relations $\geq_e$ and $\geq_s$ characterize e-positivity and Schur-positivity respectively in the following way.

Theorem 1. *Let K_n be the complete graph on n vertices. Then $G \geq_e K_n$ iff G is e-positive, and $G \geq_s K_n$ iff G is Schur positive.*

As mentioned previously, the chromatic symmetric functions of trees are of great interest. We have discovered the following properties of trees in the posets we have introduced.



(a) All 4 vertex connected graphs. Note that graph 3 is a dwarf squid.



(b) Hasse diagram of $(\mathcal{C}_4, \leq_e)$ on the left and Hasse diagram of $(\mathcal{C}_4, \leq_s)$ on the right.

Figure 2: Note that by Theorem 1, the star S_4, indicated by graph 1, is neither e-positive nor Schur positive. We can also see this by noting that $[e_{(2,2)}]X_{S_4} = -2$ and $[s_{(2,2)}]X_{S_4} = -1$.

Theorem 2. *In the posets $(\mathcal{C}_n, \leq_e)$ and $(\mathcal{C}_n, \leq_s)$, the following hold.*

(a) If T is a tree, then T is a maximal element. Hence, the set of n-vertex trees form an antichain.

(b) The star graph S_n is a minimal element.

We have also studied the chromatic symmetric functions of specific graphs, such as the lollipop graphs introduced in [1] and dwarf squids, which consist the union of a cycle and a star, with a vertex in the cycle identified with the centre of the star.

Theorem 3. *In the poset $(\mathcal{C}_n, \leq_e)$, the set of n-vertex lollipop graphs forms a chain and the set of n-vertex dwarf squids forms an antichain.*

Next, let $\alpha(G)$ denote the size of the maximum independent set of G. We have discovered the following property of the relations $\leq_e$ and $\leq_s$ pertaining to independent sets of graphs.

Theorem 4. *If $G \geq_e H$ or $G \geq_s H$, then $\alpha(G) \geq \alpha(H)$.*

As further work, we hope to prove an analogous result to Theorem 3 in the Schur poset $(\mathcal{C}_n, \leq_s)$, since computations done in the Sage computer algebra system suggest that an analogous result is true. Furthermore, we have noticed that some graphs with cycles are maximal/minimal elements, and that there are other minimal trees besides the star. As such, we hope to characterize such graphs. Finally, we hope to show the following property of the relations $\leq_e$ and $\leq_s$.

Conjecture 5. *If $G \geq_e H$ or $G \geq_s H$, then $E(G) \leq E(H)$ where $E(G)$ denotes the number of edges in graph G.*

References

[1] Samantha Dahlberg and Stephanie van Willigenburg, *Lollipop and Lariat Symmetric Functions*, `arXiv:1702.06974v1`.

[2] Richard Stanley, *A Symmetric Function Generalization of the Chromatic Polynomial of a Graph,* Adv. Math. 111 (1995), 166-194.

[3] Richard Stanley, *Enumerative Combinatorics Vol. 2.*