

**SUMMER 2017: NSERC USRA REPORT
COMPUTING SHAPE DNA USING THE CLOSEST POINT METHOD**

RACHEL HAN

1. Introduction. The Laplace–Beltrami operator is widely used in geometric modelling and computer graphics for applications such as smoothing, segmentation and registration of shapes [2, 5]. Another novel application is characterizing shapes by extracting their ‘fingerprints’. The idea of ‘Shape-DNA’ was first introduced by Reuter, Wolter and Peinecke [3]. The full spectrum of eigenvalues of Laplace–Beltrami is useful because it contains intrinsic information of Riemann manifolds, such as volume and surface area [3]. We use the smallest 50 eigenvalues, scaled by the first non-zero eigenvalue as the Shape-DNA. Since the spectrum is isometry invariant (isometries include rotation, translation and reflection), the Shape-DNA can characterize objects robustly, independent of parametrization. Storing and processing Shape-DNA enables identification in shape databases in machine learning applications.

Using the previously developed closest point method (CPM) [4, 1], we discretized the Laplace–Beltrami operator to a matrix M . Then, MATLAB was used to solve the eigenvalue problem on various surfaces to compute approximations to the Shape-DNA. We improved the memory usage of the default MATLAB ‘eigs’ algorithm using an iterative approach. Finally, we used multidimensional scaling plots to represent the similarities between the shapes.

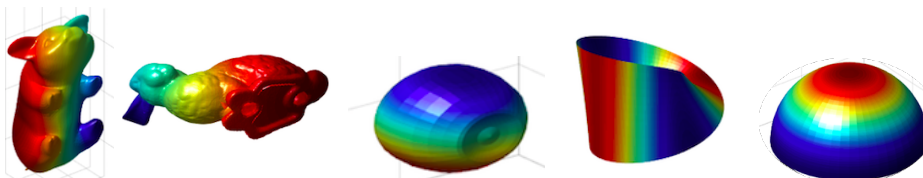
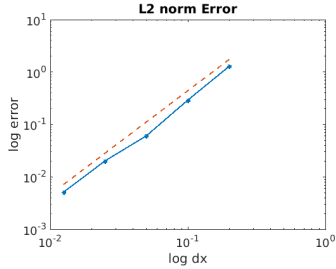
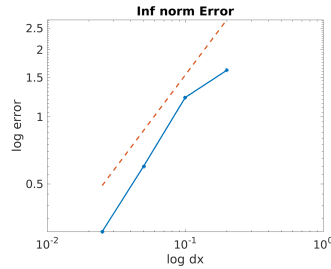


FIG. 1. *Eigenfunctions on pig, Stanford bunny, apple, mobius strip and hemisphere surfaces*

2. Implementation. The discretized L–B operator matrix M , obtained through Closest Point Method [4], is a sparse, nonsymmetric matrix. MATLAB’s ‘eigs’ uses Arnoldi iteration to iteratively compute the largest eigenvectors—starting with a random vector v , the sequence of vectors in a Krylov space $\{Mv, M^2v, \dots, M^{n-1}v\}$ converges to the eigenvector v_{max} corresponding to the largest eigenvalue. Similarly, using this iteration on the inverse of M gives eigenvectors corresponding to the smallest eigenvalue. This requires that we solve the subproblem $Mv_n = v_{n-1}$ in every step of the iteration. The default method of solving this linear system in MATLAB is the ‘\’ operator. This method does not take sufficient advantage of the sparsity of M , and runs out of memory for $\Delta x \leq 0.0125$ for a unit sphere. We implemented an iterative solver using GMRES to solve the subproblem using less memory. Incomplete L–U factorization was performed for preconditioning, which was essential for the solution to converge in a reasonable amount of time. While this method saved memory space, so far it seems to be slower than the default direct method.

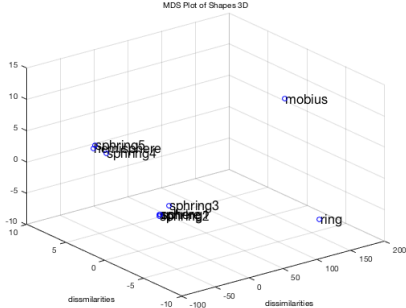
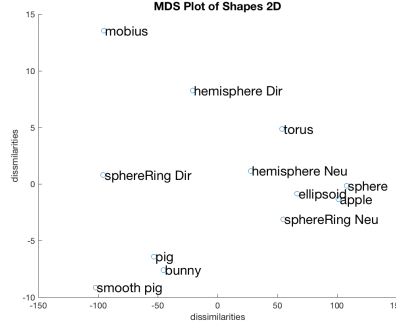
3. Numerical verification and convergence studies. The error in computing the truncated spectrum (λ_n) of the L–B operator using the closest point method should be $C(n)\Delta x^2$; second-order accuracy as $\Delta x \rightarrow 0$ was observed in practice on a unit sphere and other surfaces. The constant $C(n)$ would grow larger as n increases as

smaller Δx will be required to resolve the higher eigenvalues due to the eigenfunctions becoming more oscillatory. We rotated various surfaces in space and compared the Shape-DNA. As $\Delta x \rightarrow 0$, the difference converges at second-order (Figures 2 and 3), which confirms the rotational invariance.

FIG. 2. *Ellipsoid rotation test*FIG. 3. *Hemisphere with Neumann BC*

4. Preliminary results on clustering using multidimensional scaling.

Multidimensional scaling is an algorithm that takes vectors and clusters them through nonlinear dimension reduction. Using MATLAB’s ‘mdscale’ command, we attempted to cluster a collection of shapes using 2D and 3D plots, as shown in Figures 4 and 5. Euclidean distance (in the Shape DNA space) was used to compute similarities. We observe, for example that in Figure 4, “sphere rings” (spheres punctured by holes of various sizes) are clustered with the sphere when the hole is small, and ones with big holes are with the hemisphere, as might be expected. In Figure 5, rounded and closed shapes such as apple, sphere and ellipsoid are clustered.

FIG. 4. *Clustering punctured spheres, others*FIG. 5. *Clustering a collection of shapes*

Acknowledgments. Work funded by an Undergraduate Summer Research Award from NSERC, supervised by Dr. Colin Macdonald. Thanks to Dr. Steve Ruuth and Chingyi Tsoi at the PIMS 2017 Summer School, as well as Tony Wong and UBC Math IT.

REFERENCES

- [1] C. B. MACDONALD, J. BRANDMAN, AND S. J. RUUTH, *Solving eigenvalue problems on curved surfaces using the closest point method*, J. Comput. Phys., 230 (2011).
- [2] M. REUTER, S. BIASOTTI, G. P. D. GIORGI, AND M. SPAGNUOL, *Discrete Laplace–Beltrami operators for shape analysis and segmentation*, Computers & Graphics, 33 (2009).
- [3] M. REUTER, F.-E. WOLTER, AND N. PEINECKE, *Laplace–Beltrami spectra as ‘shape-DNA’ of surfaces and solids*, Computer-Aided Design, 38 (2006).
- [4] S. J. RUUTH AND B. MERRIMAN, *A simple embedding method for solving partial differential equations on surfaces*, J. Comput. Phys., 227 (2008).
- [5] S. SEO, M. CHUNG, AND H. VORPERIAN, *Heat kernel smoothing using Laplace–Beltrami eigenfunctions*, Medical Image Computing and Computer-Assisted Intervention, 230 (2010).