Summer 2017
NSERC USRA Report
Minimal Surfaces

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1 Introduction

This summer, I worked with Professor Ailana Fraser and studied free boundary minimal surfaces immersed in the unit ball. These surfaces have zero mean curvature everywhere and meet the ball orthogonally. Let such a surface be denoted Σ. Consider the set of conformal transformations of the unit ball, and let any surface obtained by these transformations be denoted $\gamma(\Sigma)$. The questions that we considered are:

1. Is it true that $|\gamma(\Sigma)| \leq |\Sigma|$ for 2 dimensional manifolds, with equality only when the transformation is isometric?
2. Is it true that $|\gamma(\partial\Sigma)| \leq |\partial\Sigma|$ for all dimensions, with equality only when the transformation is isometric?

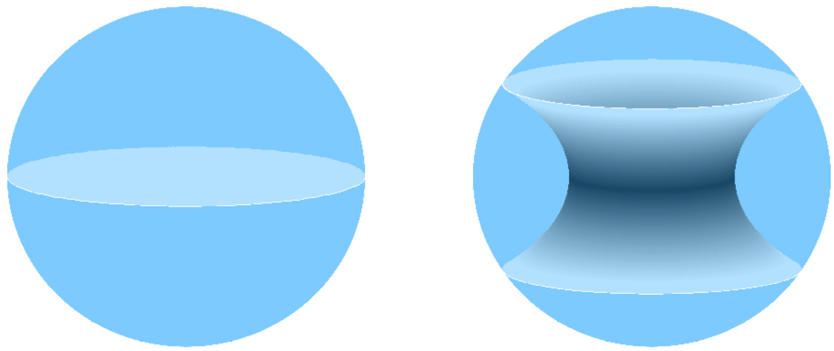
The first question, concerning the decrease in area with conformal deformations, is inspired by the use of the analogous proof for minimal surfaces immersed in the unit sphere for solving the Willmore conjecture [3]. The second question, concerning the decrease in boundary volume with conformal deformations, is proved for the 2-dimensional case in [1] and [2].

2 Area Decrease

The set of conformal transformations of the ball can be described by the following map [4]:

$$\gamma(x) = \frac{(1-|y|^2)x - (1-2y\cdot x+|x|^2)y}{1-2y\cdot x+|y|^2|x|^2}, \tag{1}$$

where $y \in B^n$ and $x \in \Sigma$. Let $y = \alpha\xi$, where $\alpha \in (0,1)$ and $|\xi| = 1$. The conformal map moves the origin to $-y$ and is the identity map when $\alpha = 0$. Let $\gamma_\xi^\alpha(\Sigma)$ denote the surface transformed according to the conformal map with $y = \alpha\xi$.



(a) Equatorial disk (b) Critical catenoid

Figure 1: Free boundary minimal surfaces in B^3 with known parametrizations.

Supporting evidence exists for $|\gamma(\Sigma)| \leq |\Sigma|$ to hold. It is proved that $|\Sigma| \geq \pi$ [1]. In the limit that α approaches 1, $|\Sigma| = \pi$ if ξ intersects the boundary of the original minimal surface and 0 otherwise. Therefore, it is known that for the one parameter family of conformal transformations obtained by fixing ξ, there is no net increase in the area from $\alpha = 0$ to $\alpha = 1$. In addition, for normal variations, the second variation is known to be negative [1]. To add to this evidence, I studied the change in area of the 2-dimensional manifolds for which we have explicit parametrizations: the equatorial disk and the critical catenoid (Fig. 1), immersed in B^3, and the critical Mobius band, immersed in B^4 [1]. This work included analytical and numerical computations.

2.1 Analytic Results

Supporting evidence for the statement, $|\gamma(\Sigma)| \leq |\Sigma|$, was obtained by considering the area after conformal deformations of the free boundary minimal surfaces with explicit parametrizations. The following results were proved analytically:

- For the equatorial disk, $|\gamma(\Sigma)| \leq |\Sigma|$ holds with equality only when $\gamma(\Sigma) = \Sigma$.
- For the critical catenoid and the critical Mobius band, the maximum values of the second variation of area at $\alpha = 0$ are negative (≈ -7.0439 for the critical catenoid and ≈ -8.9241 for the critical Mobius band). Therefore, the original surfaces, Σ, are at least local maximums of the area function among surfaces obtained by $\gamma(\Sigma)$.

2.2 Numerical Evidence

Numerical evidence suggests that the area of the critical catenoid strictly decreases for all possible transformations. For the critical catenoid, without loss of generality, we can restrict ξ to $(\xi_1, 0, \sqrt{1-\xi_1})$, where $\xi_1 \in (0,1)$.

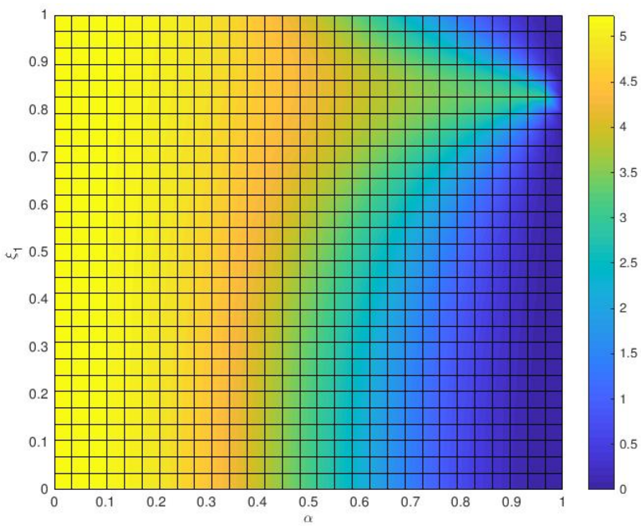
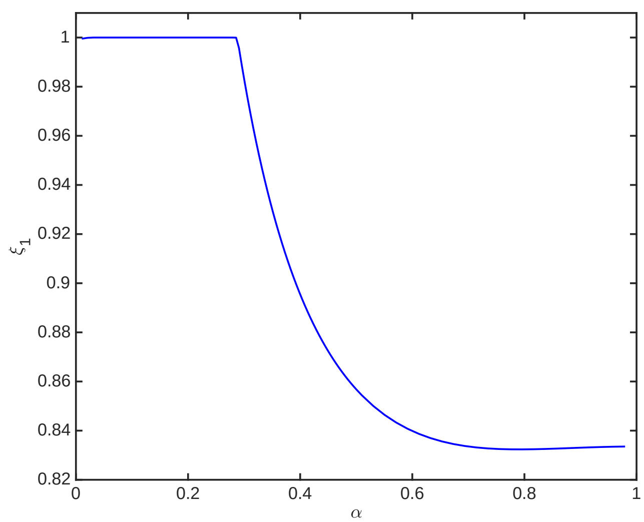


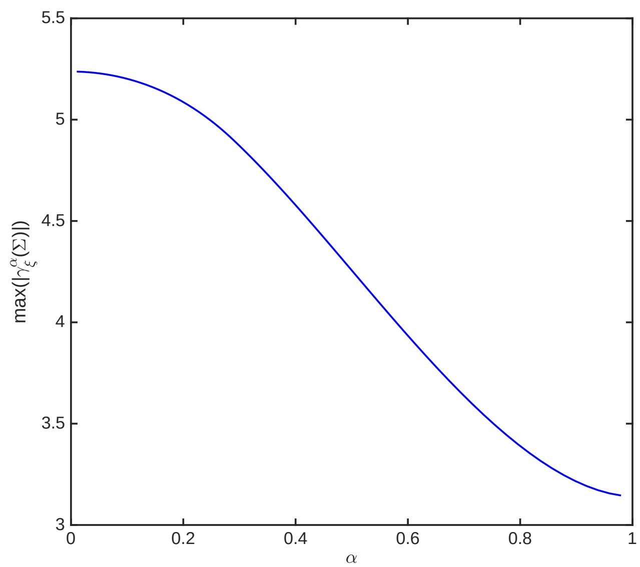
Figure 2: Numerically computed catenoid area for a grid of evenly spaced ξ_1 and α values.

Figure 2 is a plot of the area function across the (α, ξ_1) parameter space. Since the areas on this plot are computed for an evenly spaced grid of finitely many points, detail is missing in regions with large gradients.

To get a more detailed view of this behaviour, the MATLAB fmincon optimization function was used to determine the value of ξ_1 that produces the maximum area for a range of α values (Fig. 3). As expected, as α approaches 1, the ξ_1 value that maximizes the area approaches the value that causes ξ to intersect the boundary of the critical catenoid. When $\alpha = 1$, this ξ is the only value that does not result in $|\gamma(\Sigma)| = 0$. There is a clear decrease in area along the curve of maximum area for fixed values of α, indicating that it is likely that $|\gamma(\Sigma)| < |\Sigma|$ holds for all $\alpha > 0$.



(a) Values of ξ_1 that produce the surface of maximum area for fixed values of α.



(b) The change in the maximum obtainable area area with α.

Figure 3: Optimization results for the area of the transformed catenoid. The MATLAB fmincon function was used with the interior point algorithm and a specified gradient. Starting points of $\xi_1 = 0.01$ and 0.99 both produced similar curves, with a $\max|\Delta\xi_1| \approx 2.1234 \cdot 10^{-4}$ across the plotted range of α.

3 Boundary Volume Decrease

3.1 Supporting Evidence

The boundary volume of a minimal k-dimensional manifold immersed in B^n is known to decrease under conformal transformations of the unit ball for $k = 2$ [1][2]. Additionally, the second variation at $\alpha = 0$ is proved to be negative in all dimensions [1]. We found another necessary condition for the inequality concerning the boundary volume to hold and showed that it is satisfied. The condition is that for $|\gamma(\partial\Sigma)| \leq |\partial\Sigma|$ to hold, it is necessary that

$$\int_{\partial\Sigma} u^{(k-1)/2} dV_{\partial\Sigma} \leq \int_{\partial\Sigma} dV_{\partial\Sigma}, \tag{2}$$

where u^2 is the conformal factor associated with the transformation.

References

[1] A. Fraser, R. Schoen, Minimal Surfaces and Eigenvalue Problems, *Contemp. Math.* **599** (1993), 105–121.

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[3] F.C. Marques, A. Neves, Min-Max theory and the Willmore conjecture, *Ann. Math*, **179** (2016), 683-782.

[4] R. Schoen, S.T. Yau, *Lectures on Differential Geometry*, International Press, 1994.