PROPERTIES OF THE CONVOLUTION OPERATOR GENERATED BY THE CANTOR MEASURE

C. THIABAUD ENGELBRECHT AND YANZE LIU

International Student Research Report, Summer 2017

1. Introduction

This report summarises some important results found during Summer 2017, by undergraduate researchers Thiabaud Engelbrecht and Yanze "Troy" Liu, under Professor Malabika Pramanik, and her PhD student, Rob Fraser. Our research focused on exploring the properties of the convolution operator generated by the Cantor measure.

We define the Cantor measure λ to be the weak limit of $\frac{\mathbf{1}_{C_k}}{|C_k|}$, where C_k is the kth iteration of the construction of the Cantor set.

We then consider:

$$||f * \lambda||_{L^q(\mathbb{T})} \leq ||f||_{L^p(\mathbb{T})} \forall f \in L^p(\mathbb{T}). \tag{1.1}$$

We would like to know for any given q what the smallest (i.e. "best") p value that Equation 1.1 holds for is.

Oberlin has done work in this area, with two papers, [1] & [2], already published on this problem. We attempt to build on Oberlin's results. One of his more notable results is that Equation 1.2 (below) implies Equation 1.1:

$$\left(\frac{1}{3}\left(\left(\frac{a+b}{2}\right)^q + \left(\frac{a+c}{2}\right)^q + \left(\frac{b+c}{2}\right)^q\right)\right)^{1/q} \leq \left(\frac{a^p+b^p+c^p}{3}\right)^{1/p} \forall a,b,c \in \mathbb{R}^+ \tag{1.2}$$

2. Numeric Results

Using numeric methods, we managed to approximate p to 4 decimal places for various q values. A graph using this data is included below:

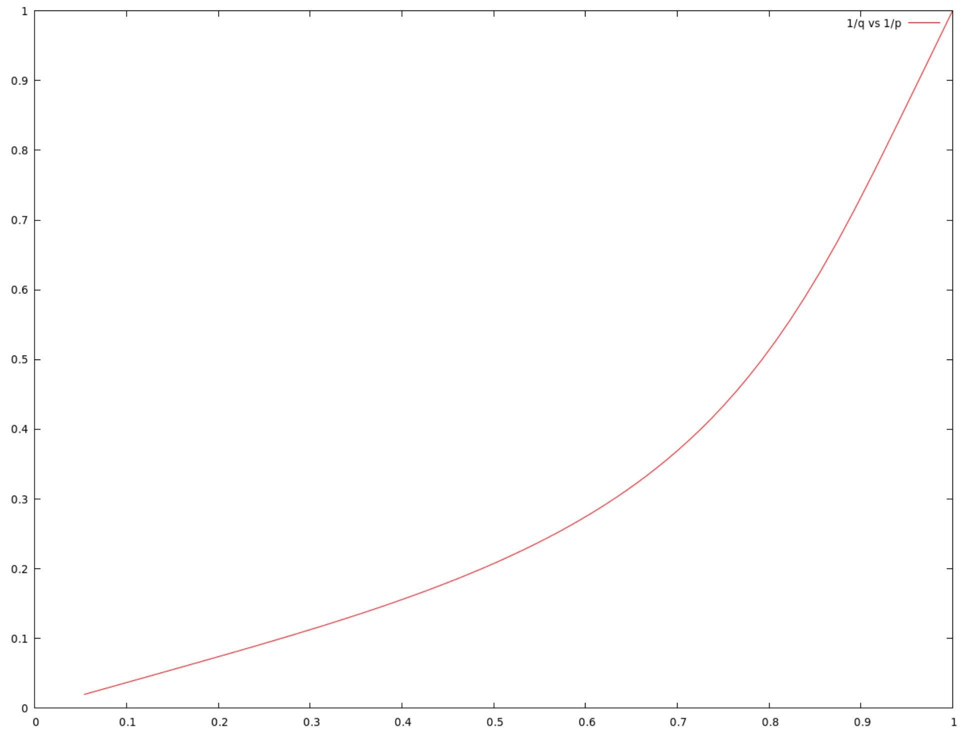

FIGURE 1. Graph of $1/q$ versus $1/p$.

3. Theoretic Results

Theorem 3.1. *Given Equation 1.2 with $q = 2$ holds for all $a, b, c \geq 0$, we will show that p can not be less than $\frac{\log 4}{\log 3}$.*

Proof. We first focus on the forward direction:

Consider the point $(a, b, c) = (1, \frac{1}{3}, \frac{1}{3})$. Plugging this point into Equation 1.2 at $q = 2$, we will get:

$$\begin{aligned}
\left[\frac{1}{3}\left(\left[\frac{2}{3}\right]^2 + \left[\frac{2}{3}\right]^2 + \left[\frac{1}{3}\right]^2\right)\right]^{\frac{1}{2}} &\leq \left[\frac{1 + 2 \cdot \left(\frac{1}{3}\right)^p}{3}\right]^{\frac{1}{p}} \\
\left(\frac{1}{3}\right)^{\frac{1}{2}} &\leq \left(3^{-1} + 2 \cdot 3^{-p-1}\right)^{\frac{1}{p}} \\
1 &\leq 3 \times \left(3^{-1} + 2 \cdot 3^{-p-1}\right)^{\frac{2}{p}} \\
1 &\leq \left(3^{-1+\frac{p}{2}} + 2 \cdot 3^{-p-1+\frac{p}{2}}\right)^{\frac{2}{p}} \\
1 &\leq \left(3^{\frac{p-2}{2}} + 2 \cdot 3^{\frac{-p-2}{2}}\right)^{\frac{2}{p}}
\end{aligned}$$

by the monotonicity of logarithm, this inequality holds if and only if the following holds:

$$0 \leq \frac{2}{p} \log \left(3^{\frac{p-2}{2}} + 2 \cdot 3^{\frac{-p-2}{2}} \right).$$

As we are only focusing on the case when $p \geq 1$, the above inequality holds if and only if

$$1 \leq 3^{\frac{p-2}{2}} + 2 \cdot 3^{\frac{-p-2}{2}}.$$

Letting $x = 3^{\frac{p}{2}}$, we have

$$\begin{aligned}
&1 \leq \frac{x}{3} + \frac{2}{3x} \\
&0 \leq x^2 - 3x + 2 \\
&0 \leq (x-1)(x-2) \\
&2 \leq x
\end{aligned}$$

The last step is due to the fact x cannot be less than 1. We can conclude that

$$\begin{aligned}
2 &\leq 3^{\frac{p}{2}} \\
\frac{\log 4}{\log 3} &\leq p.
\end{aligned}$$

□

We now consider the converse direction:

Theorem 3.2. *If $p \geq \frac{\log 4}{\log 3}$, then Equation 1.2 at $q = 2$ holds for all $a, b, c \geq 0$.*

Proof. It is suffices to show Equation 1.2 at $q = 2$ holds for all $a, b, c \geq 0$ when $p = \frac{\log 4}{\log 3}$.

To see this, consider the functions μ and f defined on the set $A = \mathbb{Z}/3\mathbb{Z}$ by

$$\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 2 \end{cases}$$

$$f(x) = \begin{cases} a & \text{if } x = 0 \\ b & \text{if } x = 1 \\ c & \text{if } x = 2 \end{cases}$$

for any $a, b, c \geq 0$. Then it is clear that Equation 1.2 at $q = 2$ holds for $a, b, c \geq 0$ if and only if $||f * \mu||_2 \leq ||f||_p$ for all f, where $|| \cdot ||$ is the norm with respect to the uniformly distributed probability measure, m, on A. Moreover, $f * \mu$ is defined below, where c is the counting measure on A.

$$f * \mu(x) = \int_A f(t)\mu(x-t)dc(t) = \sum_{t=0,1,2} f(t)\mu(x-t)$$

so

$$\begin{aligned}||f*\mu||_2 &= [\int_A |f*\mu(x)|^2 dm(x)]^{\frac{1}{2}} \\ &= \left[\sum_{x=0,1,2} \frac{1}{3}|f*\mu(x)|^2\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{3}\left(\left[\frac{a+b}{2}\right]^2 + \left[\frac{b+c}{2}\right]^2 + \left[\frac{a+c}{2}\right]^2\right)\right]^{\frac{1}{2}}\end{aligned}$$

and

$$||f||_p = \left[\int_A |f(x)|^p dm(x)\right]^{\frac{1}{p}} = \left[\sum_{x=0,1,2} \frac{1}{3}|f(x)|^p\right]^{\frac{1}{p}} = \left[\frac{a^p+b^p+c^p}{3}\right]^{\frac{1}{p}}$$

It is trivial that when $p=\infty$, $||f*\mu||_2 \leq ||f||_p$; in other words,

$$\left[\frac{1}{3}\left(\left[\frac{a+b}{2}\right]^2 + \left[\frac{b+c}{2}\right]^2 + \left[\frac{a+c}{2}\right]^2\right)\right]^{\frac{1}{2}} \leq \max(a,b,c).$$

With $||f*\mu||_2 \leq ||f||_\infty$, if we can show $||f*\mu||_2 \leq ||f||_{\frac{\log 4}{\log 3}}$, then by the Riesz-Thorin theorem: specifically, by setting $q_1 = q_2 = 2$, $p_1 = \frac{\log 4}{\log 3}, p_2 = \infty$ and by letting the operator T defined by $Tf := f*\mu$. The operator T is clearly a linear operator on $L^\infty(A)$ and $L^{\frac{\log 4}{\log 3}}(A)$, so $||f*\mu||_2 \leq ||f||_p$ for all $p \geq \frac{\log 4}{\log 3}$, as desired.

Now we focus on the case when $p = \frac{\log 4}{\log 3}$.

It suffices to show that $g(t) = \frac{[2(1+t)^2+(2t)^2]^{\frac{1}{2}}}{[1+2t^p]^{1/p}}$, $t \geq 0$ achieves its global maximum at $t=1$. (See Lemma 3.3)

It is clear that the derivative of $g(t)$ has the same sign as $s(t) = -2t^p - 2t^{p-1} + 3t + 1$, and $s(t) \leq 0$ for $t \geq 1$. What remains to show is that $g(t) \leq g(1)$ on the interval $[0,1]$.

We first consider the roots of $s(t)$, which are the roots of $g'(t)$

$$\begin{aligned}s'(t) &= -2pt^{p-1} - 2(p-1)t^{p-2} + 3 \\ s''(t) &= -2p(p-1)t^{p-2} - 2(p-1)(p-2)t^{p-3} \\ &= -2(p-1)t^{p-3}(pt + (p-2))\end{aligned}$$

$s''(t)$ has exactly one root on $[0,1]$, namely $\frac{2-p}{p}$. Therefore, $s(t)$ has at most three roots on the interval, and so does $g'(t)$. Observe that $s(1) = s(\frac{1}{3}) = 0$. In addition, since $g'(\frac{1}{3}) = s(\frac{1}{3}) = 0$, $g''(\frac{1}{3}) \approx -0.0344 < 0$, g achieves a local maximum at $t = \frac{1}{3}$. Moreover, $s'(1) \approx -0.0474 < 0$, so g achieves another local maximum at $t=1$ on $[0,1]$ because $s(t)$, thus $g'(t)$, is positive in some neighbourhood of $t=1$. Consequently, the global maximum of g on the $[0,1]$ must be achieved at $t = \frac{1}{3}$ or at $t=1$. As $g(1) = g(\frac{1}{3})$, we can conclude that g achieves its maximum at $t=1$ as desired. □

Lemma 3.3. *Let* $p = \frac{\log 4}{\log 3}$. *If* $g(t) = \frac{[2(1+t)^2+(2t)^2]^{\frac{1}{2}}}{[1+2t^p]^{1/p}}$ *achieves its global maximum at* $t=1$, *then Equation 1.2 at* $q=2$ *holds.*

Proof. Observe that Equation 1.2 at $q = 2$ holds if and only if

$$\left(\frac{1}{12}\right)^{\frac{1}{2}} 3^{\frac{1}{p}} \left(\left[\frac{a+b}{d}\right]^2 + \left[\frac{b+c}{d}\right]^2 + \left[\frac{a+c}{d}\right]^2\right)^{\frac{1}{2}} \leq 1$$

where $d = [a^p + b^p + c^p]^{\frac{1}{p}}$. Note that we are assuming that not all of $a, b, c = 0$, when they are all 0, the equality holds trivially.

By a change of variables, it is sufficient to show

$$\left(\frac{1}{12}\right)^{\frac{1}{2}} 3^{\frac{1}{p}} \left([a+b]^2 + [b+c]^2 + [a+c]^2\right)^{\frac{1}{2}} \leq 1 \qquad \text{under the constraint } a^p+b^p+c^p = 1. \tag{3.1}$$

In order to show Equation 3.1, we use Lagrange Multipliers. The maximum of the function

$$u(a, b, c) = \left([a+b]^2 + [b+c]^2 + [a+c]^2\right)^{\frac{1}{2}}$$

under the constraint $v(a, b, c) = a^p + b^p + c^p = 1$ is either achieved at the boundary or the interior. If it is not achieved at the boundary, then by introducing Lagrange's multiplier, we can easily conclude that at (a_0, b_0, c_0) where the maximum is achieved, at least two of a_0, b_0, c_0 are equal. WLOG, we assume $a_0 = b_0$. It follows that:

$$\max f = \left[(2a_0)^2 + (a_0 + c_0)^2\right]^{\frac{1}{2}} = c_0 \left[(\frac{2a_0}{c_0})^2 + (\frac{a_0}{c_0} + 1)^2\right]^{\frac{1}{2}}$$

Because that g achieves global its maximum at $t = 1$, we have that $g(\frac{a_0}{c_0}) \leq g(1)$; that is,

$$\left[(\frac{2a_0}{c_0})^2 + (\frac{a_0}{c_0} + 1)^2\right]^{\frac{1}{2}} \leq g(1) \left(2(\frac{a_0}{c_0})^p + 1\right)^{\frac{1}{p}}$$

Thus

$$\max f = c_0 \left[(\frac{2a_0}{c_0})^2 + (\frac{a_0}{c_0} + 1)^2\right]^{\frac{1}{2}} \leq c_0 g(1) \left(2(\frac{a_0}{c_0})^p + 1\right)^{\frac{1}{p}} = g(1) = 12^{\frac{1}{2}} \left(\frac{1}{3}\right)^{\frac{1}{p}}$$

Therefore,

$$\begin{aligned}
LHS\ of \text{ Equation 3.1} &= \left(\frac{1}{12}\right)^{\frac{1}{2}} 3^{\frac{1}{p}} \left([a+b]^2 + [b+c]^2 + [a+c]^2\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{12}\right)^{\frac{1}{2}} 3^{\frac{1}{p}} (\max f) \\
&\leq \left(\frac{1}{12}\right)^{\frac{1}{2}} 3^{\frac{1}{p}} g(1) \\
&= 1
\end{aligned}$$

Then we consider the case when the maximum is achieved at the boundary. WLOG we assume that $a = 0$ because of the symmetry of a, b, c. Equation 1.2 at $q = 2$ becomes

$$\left[\frac{1}{3}\left(\left[\frac{b}{2}\right]^2 + \left[\frac{b+c}{2}\right]^2 + \left[\frac{c}{2}\right]^2\right)\right]^{\frac{1}{2}} \leq \left[\frac{b^p + c^p}{3}\right]^{\frac{1}{p}} \tag{3.2}$$

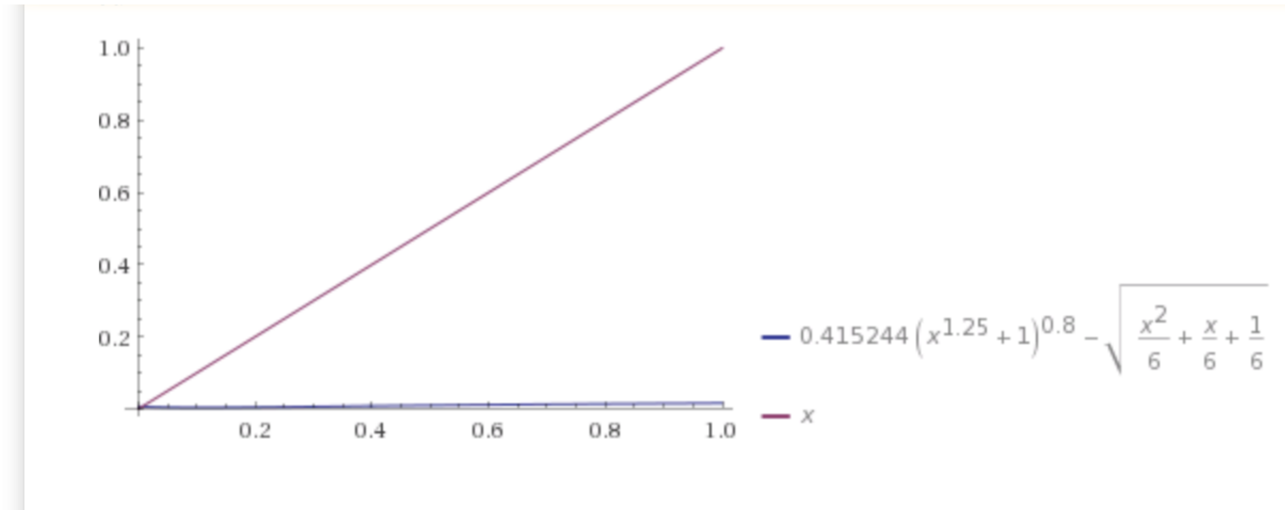

FIGURE 2. A closer look at the function.

We can assume WLOG that $b \leq c$, and divide both sides by c:

$$\left[\frac{1}{3}\left(\left[\frac{b}{2c}\right]^2+\left[\frac{\frac{b}{c}+1}{2}\right]^2+\left[\frac{1}{2}\right]^2\right)\right]^{\frac{1}{2}} \leq \left[\frac{\left(\frac{b}{c}\right)^p+1}{3}\right]^{\frac{1}{p}}$$

Let $x=\frac{b}{c}, 0 \leq x \leq 1$, what remains to show is that:

$$\left[\frac{1}{3}\left(\left[\frac{x}{2}\right]^2+\left[\frac{x+1}{2}\right]^2+\left[\frac{1}{2}\right]^2\right)\right]^{\frac{1}{2}} \leq \left[\frac{x^p+1}{3}\right]^{\frac{1}{p}}$$

This can be checked through the graph of difference between the RHS and LHS of the above inequality.

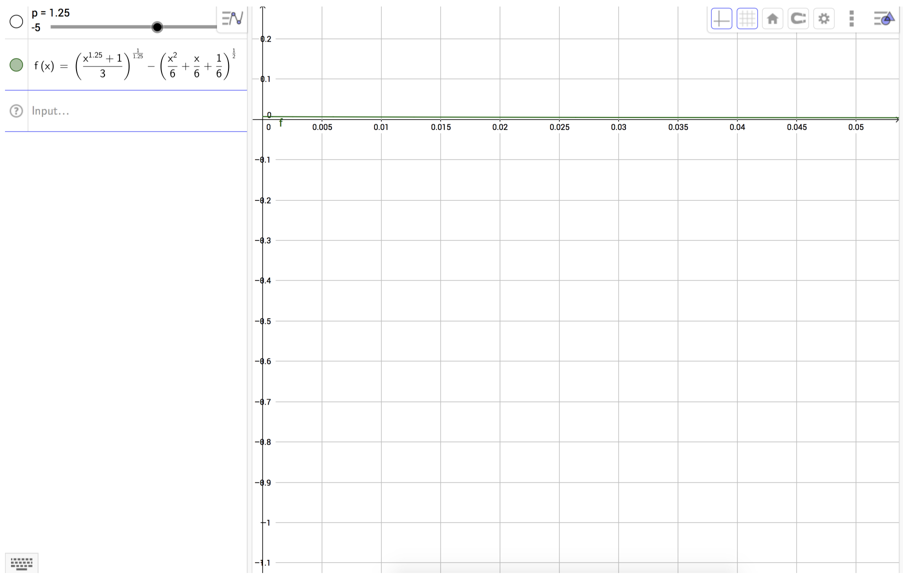

FIGURE 3

Note that in the graph, the value of p is set to be 1.25, which is smaller than $\frac{\log 4}{\log 3}$, and the inequality holds clearly for $p=\infty$; therefore by similar interpolation argument we have used previously, for any value of $p \geq 1.25$, in particularly $p = \frac{\log 4}{\log 3}$, Equation 3.2 holds. $\square$

We would like to show:

$$\left[\frac{1}{3}\left(\left[\frac{a+b}{2}\right]^3+\left[\frac{b+c}{2}\right]^3+\left[\frac{a+c}{2}\right]^3\right)\right]^{\frac{1}{3}} \leq \left[\frac{a^p+b^p+c^p}{3}\right]^{\frac{1}{p}} \tag{3.3}$$

holds for all $a, b, c \geq 0$, if and only if $p \geq \frac{3}{2}$. Note that p is always positive.

Without loss of generality, we assume that $a+b+c=3$, and write $a = 1+x, b = 1+y, c = 1+z$; thus we have $x, y, z \geq -1$ and $x+y+z=0$. Then the left hand side becomes

$$\left[\frac{1}{3}\left(\left[1+\frac{x+y}{2}\right]^3+\left[1+\frac{y+z}{2}\right]^3+\left[1+\frac{x+z}{2}\right]^3\right)\right]^{\frac{1}{3}} = \left[\frac{1}{3}\left(\left[1-\frac{x}{2}\right]^3+\left[1-\frac{y}{2}\right]^3+\left[1-\frac{z}{2}\right]^3\right)\right]^{\frac{1}{3}}.$$

What remains to show is that

$$\left[\frac{1}{3}\left(\left[1-\frac{x}{2}\right]^3+\left[1-\frac{y}{2}\right]^3+\left[1-\frac{z}{2}\right]^3\right)\right]^{\frac{1}{3}} \leq \left[\frac{(1+x)^p+(1+y)^p+(1+z)^p}{3}\right]^{\frac{1}{p}} \tag{3.4}$$

holds for all $x, y, z \geq -1$ with $x+y+z=0$ if and only if $p \geq \frac{3}{2}$.

Theorem 3.4. *Given Equation 3.4 holds for all $x, y, z \geq -1$ with $x+y+z=0$, we have $p \geq \frac{3}{2}$.*

Proof. Let $f(x,y,z)$ be the left hand side of Equation 3.4 and $g(x,y,z,p)$ be the right hand side of Equation 3.4.

$$\begin{aligned} f(x,y,z) &= \left[\frac{1}{3}\left(\left[1-\frac{x}{2}\right]^3+\left[1-\frac{y}{2}\right]^3+\left[1-\frac{z}{2}\right]^3\right)\right]^{\frac{1}{3}} \\ &= \left[1-\frac{x}{2}-\frac{y}{2}-\frac{z}{2}+\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{4}-\frac{x^3}{24}-\frac{x^3}{24}-\frac{x^3}{24}\right]^{\frac{1}{3}} \\ &= \left[1+\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{4}-\frac{x^3}{24}-\frac{y^3}{24}-\frac{z^3}{24}\right]^{\frac{1}{3}} \end{aligned}$$

Note that

$$\begin{aligned} &\frac{\partial f}{\partial x}(0,0,0)=\frac{\partial f}{\partial y}(0,0,0)=\frac{\partial f}{\partial z}(0,0,0)=0 \\ &\frac{\partial^2 f}{\partial x^2}(0,0,0)=\frac{\partial^2 f}{\partial y^2}(0,0,0)=\frac{\partial^2 f}{\partial z^2}(0,0,0)=\frac{1}{6} \\ &\frac{\partial^2 f}{\partial x\partial y}(0,0,0)=\frac{\partial^2 f}{\partial x\partial z}(0,0,0)=\frac{\partial^2 f}{\partial y\partial z}(0,0,0)=0. \end{aligned}$$

By considering the Taylor expansion of f at (0,0,0), we have

$$f(x,y,z)=1+\frac{x^2+y^2+z^2}{12}+\textit{higher order terms}.$$

Then we focus on $g(x,y,z,p)$. Let

$$k(x,y,z,p)=\left[\frac{(1+x)^p-px+(1+y)^p-py+(1+z)^p-pz}{3}\right]^{\frac{1}{p}}$$

Because $x+y+z=0$, we have $g(x,y,z,p)=k(x,y,z,p)$ Note that

$$\begin{aligned} &\frac{\partial k}{\partial x}(0,0,0,p)=\frac{\partial k}{\partial y}(0,0,0,p)=\frac{\partial k}{\partial z}(0,0,0,p)=0 \\ &\frac{\partial^2 k}{\partial x^2}(0,0,0,p)=\frac{\partial^2 k}{\partial y^2}(0,0,0,p)=\frac{\partial^2 k}{\partial z^2}(0,0,0,p)=\frac{p-1}{3} \\ &\frac{\partial^2 k}{\partial x\partial y}(0,0,0,p)=\frac{\partial^2 k}{\partial x\partial z}(0,0,0,p)=\frac{\partial^2 k}{\partial y\partial z}(0,0,0,p)=0. \end{aligned}$$

By considering the Taylor expansion of k at (0,0,0), we have

$$g(x,y,z,p)=k(x,y,z,p)=1+\frac{p-1}{6}(x^2+y^2+z^2)+\textit{higher order terms}.$$

Also, we are given that for any $x,y,z\geq -1$, $f(x,y,z)\leq g(x,y,z,p)$.

In other word,

$$1+\frac{p-1}{6}(x^2+y^2+z^2)+\textit{higher order terms}\geq 1+\frac{x^2+y^2+z^2}{12}+\textit{higher order terms}.$$

By taking $x,y,z\to 0$, we can see that $p\geq\frac{3}{2}$. □

We will start the proof of the converse direction:

Theorem 3.5. *If $p\geq\frac{3}{2}$, then Equation 3.4 holds for all $x,y,z\geq -1$ with $x+y+z=0$.*

Proof. It suffices to show Equation 3.4 holds when $p = \frac{3}{2}$; for larger p, Equation 3.4 holds by Riesz-Thorin Theorem.

Fixing $p = \frac{3}{2}$ and taking left hand side to the power of p, we will have

$$\begin{aligned}
\left[\frac{1}{3}\left(\left[1-\frac{x}{2}\right]^3+\left[1-\frac{y}{2}\right]^3+\left[1-\frac{z}{2}\right]^3\right)\right]^{\frac{1}{2}} &= \left[1-\frac{x}{2}-\frac{y}{2}-\frac{z}{2}+\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{4}-\frac{x^3}{24}-\frac{x^3}{24}-\frac{x^3}{24}\right]^{\frac{1}{2}} \\
&= \left[1+\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{4}-\frac{x^3}{24}-\frac{y^3}{24}-\frac{z^3}{24}\right]^{\frac{1}{2}} \\
&\leq 1+\frac{x^2+y^2+z^2}{8}-\frac{x^3+y^3+z^3}{48}.
\end{aligned}$$

The last inequality is due to the fact that $(1+u)^{\frac{1}{2}} \leq 1+\frac{u}{2}$ for $u \geq -1$.

Then we consider the right hand side to the power of p,

$$\frac{(1+x)^p+(1+y)^p+(1+z)^p}{3}.$$

By considering the Taylor expansion of $(1+x)^p$ at 0 ,

$$\begin{aligned}
(1+x)^p = (1+x)^{\frac{3}{2}} &= 1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16}+\frac{3x^4}{128}+\cdots \\
&= 1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16}+\frac{3}{128(c+1)^{\frac{5}{2}}}x^4 \\
&\geq 1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16},
\end{aligned}$$

for some c between 0 and x.

Therefore,

$$\begin{aligned}
\frac{(1+x)^p+(1+y)^p+(1+z)^p}{3} &\geq 1+\frac{x+y+z}{2}+\frac{x^2+y^2+z^2}{8}-\frac{x^3+y^3+z^3}{48} \\
&= 1+\frac{x^2+y^2+z^2}{8}-\frac{x^3+y^3+z^3}{48} \\
&\geq \left[\frac{1}{3}\left(\left[1-\frac{x}{2}\right]^3+\left[1-\frac{y}{2}\right]^3+\left[1-\frac{z}{2}\right]^3\right)\right]^{\frac{1}{2}}.
\end{aligned}$$

Then,

$$\left[\frac{1}{3}\left(\left[1-\frac{x}{2}\right]^3+\left[1-\frac{y}{2}\right]^3+\left[1-\frac{z}{2}\right]^3\right)\right]^{\frac{1}{3}} \leq \left[\frac{(1+x)^p+(1+y)^p+(1+z)^p}{3}\right]^{\frac{1}{p}}.$$

□

Our simplest result is the discovery of a linear lower bound for p in terms of q (note that this is not a *best* lower bound except at $q = 3$):

Theorem 3.6. $p \geq \frac{3}{4} + \frac{q}{4}$.

Proof. Use the Taylor Series of Equation 1.2, with $(a, b, c) = (1, 1, 1+x)$ up to (and including) the x^2 term. Simplify and solve for p in terms of q. □

Lemma 3.7. $\left(\frac{a^p+b^p+c^p}{3}\right)^{1/p} < \left(\frac{a^{p+\varepsilon}+b^{p+\varepsilon}+c^{p+\varepsilon}}{3}\right)^{1/p+\varepsilon}$ *for all* $\varepsilon > 0$.

Theorem 3.8. *Equation 1.2 holds with* $q = 2$ *implies* $p \geq \log_3(4)$ *is the best possible bound for* p.

Proof. Take Equation 1.2 with $q = 2$, $(a, b, c) = (1, 1, 3)$:

$$\sqrt{3} \leq \left(\frac{2 + 3^p}{3}\right)^{1/p}$$

$$3^{p/2} \leq \frac{2}{3} + \frac{3^p}{3}$$

We have a quadratic in $3^{p/2}$, which has solutions $3^{p/2} \geq 2$ and $3^{p/2} \leq 1$. The second solution is extraneous, as it gives negative values of p, while the first solution gives us $p \geq \log_3(4)$.

Next, we consider Equation 1.2 with $(p, q) = (\log_3(4), 2)$, and $(a, b, c) = (1, 1, 3)$, and see that both sides are equal.

We then apply Lemma 3.7 to show that if $p < \log_3(4)$, the equality doesn't hold. □

At this point, showing that the left side of Equation 1.2 is maximized at $(a, b, c) = (1, 1, 3)$ implies the converse of the above theorem.

Note that the proof of Theorem 3.8 is fairly general, and could be applied to various q or p values, as long as we could find the appropriate "Magic Triplet" (a, b, c) to maximise the left side of Equation 1.2.

In our attempt to find these magic triplets, we used Lagrange Multipliers and numeric (computer-assisted) methods on Equation 1.2. The (p, q) values included in Table 1 have been proven to be the best possible for the given q; other values are omitted.

TABLE 1

(p, q)	(a, b, c)
$(\log_3(4), 2)$	$(1, 1, 3)$
$(\frac{3}{2}, 3)$	$(1, 1, 1)$
$(\log_3(4), 2)'$	$(1, 1, \frac{1}{2})$

Note that $(1, 1, 1)$ will always make both sides of Equation 1.2 equal, however in the case $(p, q) = (\frac{3}{2}, 3)$, it is the only triplet that makes them equal.

4. CONJECTURES

Based on the numeric results, we conjecture that the magic triplets will always take the form (a, a, c). We also conjecture that for $q > 3$, $c/a < 1$, and for $q < 3$, $c/a > 1$. Based on this conjecture, we used Lagrange Multipliers on Equation 1.2 and found that

$$\left(\frac{1 + x}{2}\right)^{q-1} \cdot (2 \cdot x^{1-p} - 1) = 1, x \in (0, \infty) \tag{4.1}$$

where $x = c/a$.

The left side goes to ∞ as $x \to \infty$, and is negative as $x \to 0$, so this expression must have a solution as for every (p, q) by Intermediate Value Theorem.

The expression gives us the correct values of x for all the known (p, q) pairs. From here, we would like to focus on finding more q, x pairs numerically, as doing so will allow us to find the correct p value, and also prove that it is the best p value for the given q.

References

1. D. M. Oberlin, *A Convolution Property of the Cantor-Lebesgue Measure.* Colloquium Mathematicum **47** (1982), 113–117.
2. D. M. Oberlin, *A Convolution Property of the Cantor-Lebesgue Measure, II.* Colloquium Mathematicum **97** (2003), 23–28.