2017 NSERC USRA Report: Multivalued Matrices and Forbidden Configurations

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1 Introduction

Define an r-matrix to be a matrix with entries in $\{0, 1, \cdots, r-1\}$. We say an r-matrix A is simple if it has no repeated columns. Given a smaller r-matrix (configuration) F, A is said to contain F, written $F \prec A$ is a submatrix of A is equal to a row and column permutation of F. Otherwise A is said to avoid F. If $\mathcal{F}$ is a family of configurations, A avoids $\mathcal{F}$. Letting $||A||$ be the number of columns in A, we define the following functions:

$$\text{Avoid}(m, r, \mathcal{F}) = \{A : A \text{ is an } m\text{-rowed, simple } r\text{-matrix, and } \mathcal{F} \not\prec A\}$$
$$\text{forb}(m, r, \mathcal{F}) = \max\{||A|| : A \in \text{Avoid}(m, r, \mathcal{F})\}$$

When $r = 2$, we often write Avoid$(m, \mathcal{F})$ and forb$(m, \mathcal{F})$ in place of Avoid$(m, 2, \mathcal{F})$ and forb$(m, 2, \mathcal{F})$. Also, if $\mathcal{F} = \{F\}$, we typically write Avoid(m, r, F) and forb(m, r, F). forb(m, F) typically has polynomial growth rate in m, and determining the growth rate of forb$(m, \mathcal{F})$ for various $\mathcal{F}$ is a central problem in the field of forbidden configurations.

We will now define a family of forbidden configurations of particular interest. Define $I_\ell(a, b)$ to be the ℓ matrix with a's on the diagonal and b's off the diagonal. $T_\ell(a, b)$ to be the $\ell \times \ell$ matrix with a's on and below the diagonal and b's above the diagonal. Define $\mathcal{T}_\ell(r) = \{T_\ell(a, b) : a, b < r,\ a \neq b\} \cup \{I_\ell(a, b) : a, b < r,\ a \neq b\}$. The following res.ult is due to Anstee and Lu [2]:

Theorem 1.1 *forb* $(m, r, \mathcal{T}_\ell(r)) \leq 2^{c\ell^2}$

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where c is a constant that depends on r. This result is remarkable since the bound is a constant independent of m.

A question one may ask is what would happen if the 0-1 matrices were removed from $\mathcal{T}_\ell(r)$; that is, if we forbid $\mathcal{T}_\ell(r)\backslash\mathcal{T}_\ell(2)$. Since there are no 0-1 matrices in this forbidden family, $\text{forb}(m, f, \mathcal{T}_\ell(r)\backslash\mathcal{T}_\ell(2)) \geq 2^m$, as taking all possible 0-1 columns avoids $\mathcal{T}_\ell(r)\backslash\mathcal{T}_\ell(2)$. Is this construction the best possible? The answer is yes, asymptotically:

Theorem 1.2 *forb*$(m, r, \mathcal{T}_\ell(r)\backslash\mathcal{T}_\ell(2))$ *is* $\Theta(2^m)$

One can think of forbidding $\mathcal{T}_\ell(r)\backslash\mathcal{T}_\ell(2)$ as turning an r-matrix into a 0-1 matrix. If this is the case, then one would expect that $\text{forb}(m, r, \mathcal{T}_\ell(r)\backslash\mathcal{T}_\ell(2) \cup \mathcal{F}) = \Theta(\text{forb}(m, \mathcal{F}))$ for any forbidden 0-1 family $\mathcal{F}$. Determining whether this holds was the central question of this summer's project. We focused on the case where $\mathcal{F} = \{F\}$.

2 Results

In the course of our summer we focused exclusively on the case $r = 3$, as doing so greatly simplified our arguments. We will assume $r = 3$ throughout the rest of this report. The justification for this narrowness is given by the following result which we proved:

Theorem 2.1 *There exists a function* $f(\ell)$ *such that forb*$(m, r, \mathcal{T}_\ell(r)\backslash\mathcal{T}_\ell(2) \cup \mathcal{F}) = \Theta($*forb*$(m, r, \mathcal{T}_{f(\ell)}(3)\backslash\mathcal{T}_{f(\ell)}(2) \cup \mathcal{F}$ *for all forbidden 0-1 families* $\mathcal{F}$*.*

Hence, solving the problem for $r = 3$ also solves it for all $r > 3$.

Most of our results are for $\mathcal{F} = \{F\}$ where F is a two-columned matrix. The asymptotics of $\text{forb}(m, F)$ for two-columned F were completely determined in ??. We define some notation: let $F_{a,b,c,d}$ be the matrix with a rows of [0 0], b rows of [0 1], c rows of [1 0] and d rows of [1 1]. Note that by the definition of avoidance, row order is irrelevant. In this notation, for example, the identity I_2 is $F_{0,1,1,0}$. Define $\text{forb}_k(m, r, \mathcal{F})$ to be the maximum number of columns in an m-rowed r-matrix avoiding $\mathcal{F}$ where every column has exactly k 0's. The main result for two-columned matrices is as follows:

Theorem 2.2 *If* F *has two columns, then forb*$(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup F)$ *is* $\Theta\left(\sum_{k=0}^{m} \text{forb}_k(m, F)\right.$.

This theorem gives the correct bound for many two-columned matrices, including $F_{0,b,b,0}$, $F_{1,b+1,b+1,1}$ - here b is any positive integer. For $F_{1,1,1,1}$, our theorem gives a bound of $n \log n$, which is too high by a factor of $\log n$. The theorem fails to resolve the case of $F_{0,b,b+1,0}$, although it should be noted that since $\text{forb}_k(m, F) \leq \text{forb}(m, F)$, $\text{forb}(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup F)$ is $\Theta(m \cdot \text{forb}(m, F))$ for all two-columned F.

We also have a general result which is particularly applicable in the two-columned case. Given a configuration F, define F_{01} to be F with a row of 1's and a row of 0's added.

Theorem 2.3 *$forb(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup F_{01})$ is $O(m \cdot forb(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup F)$.*

This is consistent with the analogous result in the 0-1 case. In particular, this gives the correct bound for $F_{a,b+1,b+1,a}$, for arbitrary $a, b > 0$. Based on the bounds given in [1], proving a bound for $F_{0,1,1,0}$ and $F_{0,b,b+1,0}$ consistent with the 0-1 case would imply that $\text{forb}(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup F)$ is $\Theta(\text{forb}(m, F))$ for all 2-columned F.

Although most of our work was focused on the two-columned case, a few results were found for other matrices. Define $t \times F$ to be t copies of F concatenated horizontally. Then the following holds:

Theorem 2.4 *If F has k rows, then $forb(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup t \times F)$ is $O\left(\max\{m^k, forb(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup F)\}\right)$.*

We were able to solve the case of one particular three-columned matrix. Let $H = \begin{bmatrix} 110 \\ 011 \end{bmatrix}$.

Theorem 2.5 *$forb(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup H)$ is $O(m)$.*

Again, this is consistent with the corresponding result in the 0-1 case. Combining with the previous theorem gives the following:

Theorem 2.6 *$forb(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup H)$ is $O(m^2)$.*

This solves the problem for many two-rowed matrices, but not all; in particular, the bounds for configurations with four distinct columns are still unknown in the multivalued case.

3 Open problems

As previously mentioned, $F_{0,b+1,b,0}$ and $F_{1,1,1,1}$ are the open problems for the two-columned case. If we can show that these bounds match the bounds in the 0-1 case, this will answer the problem in the affirmative for all two-columned matrices. In particular, we have an almost correct bound for $F_{1,1,1,1}$:

Theorem 3.1 *$forb(m, 3, \mathcal{T}_\ell(3)\backslash\mathcal{T}_\ell(2) \cup F_{1,1,1,1}$ is $O(m \log m)$*

This result is in fact a direct consequence of Theorem 2.2. All known bounds for forbidden configurations are either polynomial or at least exponential, so this result is strong evidence that the growth rate is, in fact, exactly linear, but we have no proof of this. We do not know how to approach of $F_{0,b+1,b,0}$, as Theorem 2.2 is not useful here.

Let K_k be the $k \times 2^k$ matrix of all columns on n rows. Finding the bound for K_k is of great interest. K_1 is easy, but the answer for K_2 is not yet known.

References

[1] R.P. Anstee and P. Keevash, Pairwise intersections and forbidden configurations, *European J. of Combin.* **27** (2006), 1235–1248.

[2] R.P. Anstee and Linyuan Lu, Multicoloured Families of Configurations, arXiv 1409.4123, 16pp.