

Summer 2014 - NSERC USRA Report

Eigenfunctions of the Laplacian on Triangles

Foster Tom

This summer I worked with Dr. Lior Silberman on the Hot Spots Conjecture. In plain terms, the conjecture claims that an insulated piece of metal given an initial heat distribution will eventually have its hottest point on its boundary. Formally, on a two-dimensional bounded connected domain the first Neumann eigenfunction of the Laplacian attains its extrema only on its boundary. The Polymath Project aims to prove the Hot Spots Conjecture for acute triangles. My research consists of two parts: numerically approximating these eigenfunctions for many triangles and aiming to analytically prove that the calculations sufficiently resemble the true eigenfunctions, then a stability argument of Terence Tao would address perturbation of the domain, showing that only finitely many triangles need to be considered.

Numerics

We use a finite-element method of constructing our function from exact eigenfunctions on wedges. A finite circular arc of angle $\frac{\pi}{\nu}$ with Neumann conditions imposed on the straight edges but no boundary condition on the arc has Laplacian eigenfunction corresponding to eigenvalue λ of

$$u(r, \theta) = \sum_{k=0}^{\infty} a_k J_{k\nu}(\lambda r) \cos(k\nu\theta),$$

where $J_\alpha(x)$ is the Bessel function of the first kind, one solution to the differential equation $x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$. Note that we have infinitely many arbitrary coefficients a_k because no boundary condition has been imposed on the arc.

The numerical scheme asserts that the desired function be a truncated series of this form when expressed in polar coordinates about any of the three corners; this produces linear relationships between the coefficients, then approximately solving the overdetermined system will determine these coefficients. The result is three approximate eigenfunctions u_1, u_2, u_3 corresponding to the three corners that satisfy the differential equation

exactly, satisfy the Neumann condition on the neighbouring sides exactly, and approximately satisfy the boundary condition on the opposite side of the triangle.

Analytics

We now aim to use these three functions to argue that the true eigenfunction exhibits the maximum property.

Lemma 0.1 *Suppose u satisfies the Neumann boundary conditions and $\|\Delta u - \lambda u\|_2 \leq \epsilon \|u\|_2$. Then λ is within ϵ of the nearest eigenvalue.*

Proof: The Laplacian is self-adjoint so has an orthogonal basis of eigenvectors; therefore, $\Delta - \lambda$ is a diagonal matrix multiplying each component of u by at least the distance of λ to the spectrum. ■

A similar argument of projecting u onto the nearest eigenfunction will bound the distance from u to the true eigenfunction. Therefore, we seek such a function u . Originally, we tried appending a nearly-harmonic correction to u_1 to satisfy the boundary condition on the opposite side and bounding the correction by considering the conformal equivalence of the triangle to a disk, but since the mapping is singular at the corners we were unsuccessful.

Terence Tao suggested a partition of unity, taking advantage of having three functions: For $i = 1, 2, 3$ corresponding to the three corners, let R_i be the distance to the orthocenter, χ_i be a radial function that begins at 1 and decays quickly to 0 after R_i , and let $\varphi_i = \frac{\chi_i}{\sum_j \chi_j}$, then take $u = \sum_i \varphi_i u_i$.

First note that by our choice of R_i , $\sum_j \chi_j \geq 1$ and u is well-defined. Also, each u_i has no contribution on the opposite side of the triangle, so the Neumann boundary conditions are exactly satisfied. It is convenient that $\sum_i \varphi_i$ is the constant 1. Now:

$$\|\Delta u - \lambda u\|_2 = \left\| \sum_i \Delta \varphi_i u_i + \nabla \varphi_i * \nabla u_i + \varphi_i (\Delta - \lambda) u_i \right\|_2 = \left\| \sum_{i=2}^3 \Delta \varphi_i (u_i - u_1) + \nabla \varphi_i * \nabla (u_i - u_1) \right\|_2.$$

Therefore the error depends on quantities such as $\int (u_2 - u_1) dA$, which are easily evaluated numerically. Unfortunately, it has been difficult to rigorously calculate these, as the error depends on the absolute maximum of the second derivative of the integrand. However, once formal error estimates can be established, we should be able to run our argument. Observed accuracy of our u_i in this sense, as well as to exact eigenfunctions in special cases, lend confidence to our scheme.