SUMMER 2014 - NSERC USRA REPORT: STABILITY OF SOLITONS

KYLE WAMER, SAM THRASHER

Over the past sixteen weeks, the authors have been fortunate enough to work on a project proposed and lead by Dr. Stephen Gustafson of the University of British Columbia. Entitled *Stability of Solitons*, the project's primary objective was to study the orbital stability of solitary wave solutions of certain nonlinear differential equations.

1. Introduction

Throughout, let $u = u(t, x)$ be a function with $(t, x) \in \mathbb{R} \times \mathbb{R}$.

The two nonlinear equations that were of most interest to us in this project were the Korteweg-de Vries Equation (KdV),

$$u_t + (u_{xx} + u^2)_x = 0 \tag{1}$$

and the focusing cubic nonlinear Schrödinger equation,

$$iu_t + u_x x + |u|^2 u = 0. \tag{2}$$

In addition, a cubic-quintic nonlinear Schrödinger equation, in which a quintic term $-|u|^4 u$ is added to (2), was studied to some extent. Each of these equations gives rise to solitary wave solutions [5], or *solitons*, which take the form $u(t, x) = Q(x-ct)$ or $u(t, x) = e^{i\omega t}Q(x-ct)$ in the case of (1) and (2) respectively. For a specific solitary wave profile Q, we define the family of solitons Σ_Q to be the set of spatial translates (and phase rotations, in the case of (2)) of Q. While an individual soliton solution is not stable, it is sometimes possible to establish the *orbital stability* of Σ_Q, which is defined as follows:

Definition. *Let $Q(t, x) \in \Sigma_Q$ be a solitary wave, and let $u(t, x)$ be another solution. Then Q is* ***orbitally stable with respect to the norm*** $\|\cdot\|_X$*, if $\forall\epsilon > 0, \exists\delta > 0$ such that*

$$\|u(0, x) - Q(0, x)\|_X < \delta \implies \inf_{\tilde{Q}\in\Sigma_Q} \|\tilde{Q}(t, x) - u(t, x)\|_X \leq \epsilon, \forall t > 0. \tag{3}$$

Remark: The above definition is norm-dependent, meaning orbital stability with respect to some norm $\|\cdot\|_a$ tells us no stability information with respect to some other norm $\|\cdot\|_b$. This follows from the fact that norms are not equivalent on infinite dimensional metric spaces.

With this in mind, preliminary investigations were carried out using numerical simulations which modelled small perturbations of solitons in various norms. Using an operator splitting algorithm (see [2]), we were able to look at long-time behaviour of the cubic (2) and the cubic-quintic Schrödinger equations, gathering ample evidence of $H^{\frac{1}{2}}(\mathbb{R})$ stability of the former, and $L^2(\mathbb{R})$ stability of the latter. These results motivated us to turn from simulations to analysis, in an effort to establish an orbital stability proof by the end of the summer term.

2. Results

A fundamental difference between (2) and the Cubic-Quitnic has to do with the notion of *complete integrability*. A partial explanation of this concept is that a completely integrable system gives rise to an infinite number of conserved quantities. This turns out to be extremely useful, and the majority of stability proofs of completely integrable systems exploit this feature (for instance, see [1], [3], [4]). We attempted to generalize the Bäcklund transformation associated with (2), but did not have any success.

We then considered the orbital stability of periodic travelling waves of the KdV equation. These periodic waves have a close relation to solitons: as the period approaches ∞, the periodic waves become indistinguishable in shape from solitons. In our research, we found the result [1], which guarantees the stability in the space L^2 of KdV solitons by using a simple integrable transformation known as the Gardner transformation. We are currently finishing a result which would ensure the stability in L^2 of some KdV cnoidal waves via the same transformation. That is, we aim to prove the following theorem:

Theorem. *There exist parameters $\alpha_0, A_0 > 0$ such that the following holds: Let $u_0 \in L^2_{per}(\mathbb{R})$, and let there be some $\alpha \in (0, \alpha_0)$ such that*

$$\|u_0 - Q\|_{L^2_{per}(\mathbb{R})} \leq \alpha. \tag{4}$$

Then, there exists an $x(t)$ such that the solution $u(t) \in L^2_{per}(\mathbb{R})$ of (1) with initial data u_0 satisfies orbital stability:

$$\sup_{t\geq 0} \|u(t) - Q(\cdot - x(t))\|_{L^2_{per}(\mathbb{R})} \leq A_0\alpha. \tag{5}$$

Note: We use the notation L^2_{per} to mean the space of square-integrable functions with period P:

$$L^2_{\text{per}}(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : u(\cdot, x) = u(\cdot, x + P)\}$$

Its norm is the same as the standard L^2-norm, only restricted to $(0, P]$.

References

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