# SUMMER 2014 – NSERC USRA REPORT
# *abc* TRIPLES

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Over the summer, I worked under the supervision of Professor Greg Martin and collaborated with him to write an expository paper on the $abc$ triples, from the famous $abc$ conjecture in number theory. Below is the introductory section from our joint paper:

## 1. INTRODUCTION

A, B, C... these three trite letters of the alphabet have never held a connection to something of such complexity and cardinal significance as they do here in mathematics. The *$abc$ conjecture*, so aptly named, is a simple-to-state yet challenging problem in number theory that has stumped mathematicians for the past 30 years. It has become known for its staggering number of profound implications in number theory and particularly in Diophantine equations; among this myriad of consequences are (the asymptotic version of) Fermat's Last Theorem, Mordell's conjecture [2], and Roth's theorem [1], just to name a few! (See [3] for a comprehensive list.) This conjecture is deeply intriguing for it unveils some delicate tension between the additive and multiplicative properties of integers, the bread and butter of number theorists.

We begin with a preliminary definition: the *radical* of an integer $n$, denoted by $R(n)$, is the product of all the distinct prime factors of $n$. For example, $112 = 2^4 \cdot 7$ and so $R(112) = 2 \cdot 7 = 14$. In other words, $R(n)$ is the largest squarefree divisor of $n$. Clearly, the radical is multiplicative, meaning that for pairwise relatively prime integers $a$, $b$ and $c$, we have $R(abc) = R(a)R(b)R(c)$. It is also worthwhile to point out the slight variance between a set of integers being *relatively prime* and being *pairwise relatively prime*: a set is relatively prime if there is no common prime factor amongst its elements, while pairwise relatively prime means that any two chosen integers from the set are coprime. Fortunately in our case, the ambitious assertion of the $abc$ deals only with three positive integers related by the equation $a + b = c$, which ensures that any relatively prime set $(a, b, c)$ must also be pairwise relatively prime.

***abc* Conjecture, Version 1.** *For every $\varepsilon > 0$, there exist only finitely many triples $(a, b, c)$ of relatively prime positive integers satisfying $a + b = c$ for which*

$$R(abc) < c^{1-\varepsilon}.$$

Loosely put, the radical is normally larger than $c$, yet there are occurrences, albeit limited, where $c$ is in fact the greater of the two. These are special cases in which all three of $a, b$ and $c$ are highly factorizable, say, as high powers of small prime factors, hence giving rise to a relatively small radical $R(abc)$. A simple and the smallest such example is $(a, b, c) = (1, 8, 9)$, for which $R(abc) = R(36) = 6 < 9$; these special cases are referred to as "$abc$ triples". Furthermore, one can even construct an infinite sequence of $abc$ triples! One such example is $(a, b, c) = (1, 9^n - 1, 9^n)$: it can be easily shown that $2^3$ divides $9^n - 1$ for every positive integer $n$, and so we write $b = 2^3 \cdot k$ for some positive integer $k$ and see that $R(abc)$ is at most $2 \cdot k \cdot 3 = 6k$, which is less than $c = 8k + 1$ for any $n$. We call this an "infinite family" of triples and many more will be seen later on in Section 3. Still, the norm remains that at least one out of $a, b$ and $c$ will have large prime factors; case in point, the infinite sequence $(a, b, c) = (1, 2^n - 1, 2^n)$ contains the Mersenne numbers as $b$, which is known to be squarefree whenever $n$ is prime (and may even be prime itself infinitely many times!), hence $R(b)$ here is usually quite large.

As often in literature we find various versions of the conjecture, we name a few now and number them for easy reference throughout the paper. For starters, the conjecture is commonly stated with the epsilon on the opposite side:

***abc* Conjecture, Version 2.** *For every $\varepsilon > 0$, there exist only finitely many triples $(a, b, c)$ of relatively prime positive integers satisfying $a + b = c$ for which*

$$c > R(abc)^{1+\varepsilon}.$$

Version 1 and Version 2 can be effortlessly obtained from one another; for instance, by taking the inequality in Version 1 with $\varepsilon$ and raising both sides to $1/(1-\varepsilon)$, Version 2 easily follows with the new epsilon $\varepsilon' = \frac{\varepsilon}{1-\varepsilon}$. Let us keep in mind that Version 2 has the bonus of its smooth transition to arriving at the "quality" of an $abc$ triple – a term we will define shortly in Section 2.

Because only finitely many triples exist for each $\varepsilon$, there is an eventual maximum bound on how large $c$ can be amongst these triples, and so we can state an equivalent formulation of the conjecture:

***abc* Conjecture, Version 3.** *For every $\varepsilon > 0$ there exists a positive constant $K(\varepsilon)$ such that all triples $(a, b, c)$ of relatively prime positive integers with $a + b = c$ satisfy*

$$c \leq K(\varepsilon)R(abc)^{1+\varepsilon}.$$

Indeed, for any fixed $\varepsilon$, setting that constant $K(\varepsilon)$ as the $c$ from the largest triple will suffice. And, as Version 3 is to Version 2, Version 4 is to Version 1:

***abc* Conjecture, Version 4.** *For every $\varepsilon > 0$ there exists a positive constant $K'(\varepsilon)$ such that all triples $(a, b, c)$ of relatively prime positive integers with $a + b = c$ satisfy*

$$R(abc) \geq K'(\varepsilon)c^{1-\varepsilon}.$$

Note that every condition and variable that piece together this conjecture is indispensable. For instance, without the key coprimality condition, one may write $(a, b, c) = (pa', pb', pc')$, where $p$ is a common prime factor shared by the integers, and raise $p$ to higher powers to create new triples of integers; we are then left with the trivial case where $c$ grows ad infinitum while $R(abc)$ remains unchanged. Likewise, the epsilon in the conjecture may seem like a nuisance but it is an absolute necessity. In fact, the naive assertion that $c$ can only be greater than the radical in finitely many cases, or equivalently, that there exists some absolute positive constant $\delta$ such that $R(abc) \geq \delta c$ always, is known to be false. (It is instructive to construct counterexamples to disprove this simplistic thinking and we will devote Section 3 to doing so.)

In this article we explore the $abc$ conjecture in depth and by keeping the technical prerequisites to a minimum, we hope to engage even the non-specialists in this tremendous problem in number theory. We start off in Section 2 by looking at some numerical examples of $abc$ triples that have been garnered over the years and by examining various computational techniques of obtaining such triples, before moving on to Section 3 where we shift our attention to the aforementioned infinite families of triples. We then delve into the motivation behind this deep conjecture and present some of its refinements and generalizations. Lastly, we cap off with a discussion on the progress towards and the current status of the $abc$ conjecture.

## References


[1] E. Bombieri, Roth's theorem and the abc conjecture, preprint (1994).

[2] N. Elkies, $ABC$ implies Mordell, *Int. Math. Res. Notices* (1991), no.**7**, 99–109.

[3] A. Nitaj, http://www.math.unicaen.fr/∼nitaj/abc.html#Consequences, 2013. (accessed May 2, 2014)