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Using Lie algebra Properties and Differential Equations to Solve Equations in Lie Groups

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It is a well known fact that if an element of a Lie Algebra can be expressed as a commutator of two other elements, then in the infinite term limit, its exponential (the corresponding element of the Lie Group) can also be expressed as a product of the exponentials of those elements. This project looks at a generalization of the finite term case.

Lie groups are often used to represent symmetries in the solving of differential equations. By showing that one potential symmetry can be represented in terms of the others, one can establish that if an equation satisfies those symmetries, it will also satisfy the other.

Additionally, this can be used in problems of control theory. Lie Groups can be used to represent spaces of transformations. If an apparatus has mechanisms for moving in a limited set of ways, other transformations can potentially be generated by composing these motions in certain ways. This method allows one to determine both whether this is possible in a given number of steps and to determine an explicit composition.

The method works by assuming that there are paths from the origin to the desired point in the Lie Algebra through the space of points that can be expressed in this way and that the decomposition is continuous and differentiable. More specifically, the version developed requires that, if the desired point is $e^{\epsilon A}$ for some A in the Lie Algebra, then the decomposition of e^{fA} for $f \in [0, \epsilon]$ must be both existent, continuous and differentiable. This is reasonable, given that a high enough number of steps is allowed, as if $e^{\eta_1 A}, e^{\eta_2 A}$ are decomposable, so too is $e^{(\eta_1 + \eta_2)A}$, but with a higher number of steps, by merely appending the two decompositions.

The first step of the method is to choose a basis for the Lie Algebra that includes the two elements you want to express things in terms of. We call this basis $\mathcal{B} = \{B_1, B_2, B_3, ...\}$. Next, we use the power series of the exponential to find T_{ij} in the Lie Algebra that satisfies expressions of the form $e^{B_i} B_j = T_{ij} e^{B_i}$.

Now, if R is the element whose exponentials one wishes to express in terms of the exponentials of B_1, B_2, they take the equation $e^{\epsilon R} = e^{a_1(\epsilon)B_1} e^{a_2(\epsilon)B_2} e^{a_3(\epsilon)B_1} e^{a_4(\epsilon)B_2} ... e^{a_n(\epsilon)B_n \mod 2}$, where n is the number of terms desired in the decomposition. Next, one differentiates both sides with respect to ϵ. As a Lie Algebra is a differentiable manifold, and as it was assumed that the decomposition was differentiable, this is possible. Moving the terms from the Lie Algebra to the left of the right hand side using the results $e^{B_i} B_j = T_{ij} e^{B_i}$ and cancelling the $e^{\epsilon R}$ on both sides gives us a differential equation in the Lie Algebra which can be now solved.

This project began with an evaluation of parallel parking in a simplified driving model called the unicycle model. Under the unicycle model, cars may preform two actions: They may translate (a possibly negative amount) in a straight line in the direction they are facing and they may rotate their facing direction counterclockwise (by a possibly negative amount). To parallel park, a car must translate in the perpendicular direction of its facing via a composition of these motions. We evaluate this in three ways: a simple geometric method, a more traditional algebraic method and our new method. The three ways agree on the correct solution.

Next, we use the complete classification of three-dimensional Lie Algebras given by Libor Šnobl and Pavel Winternitz in *Classification and Identification of Lie Algebras*. In all but two algebras, there was a pair of elements that generated the rest of the Algebra. In all but one case, it was possible to choose such a pair so that they were both elements of the canonical basis given by Šnobl and Winternitz. Where possible, we then looked at how to generate an arbitrary element of the Lie Group (expressed as $e^{\epsilon(\alpha X + \beta Y + \geq Z)}$, where $\{X, Y, Z\}$ was the basis) as a four-term product of exponentials of this pair. (One more term than the dimension of the Algebra gave one degree of freedom to the differential equations and thus made it so solutions generally existed) In all but one case, an explicit solution to the resulting differential equations was found.

We also investigated two special cases of interest in each Algebra: the expansion of the third basis vector in terms of the other two and the expansion of the commutator of the chosen two elements.

The one remaining case where the differential equations were in general unsolved was so_3. However, a formula for the third basis vector as a three-term expansion of the other two is known in this case, and it was checked and confirmed as a solution of the differential equations.

Finally, to show that the method works more generally, we solve one example where the Lie Algebra is four-dimensional and the desired element is not spanned by the two basis elements and their commutator, but rather obtained taking the commutator twice.