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Topology of Modular Links
Alexandre Brandts-Longtin

This summer I worked with Lior Silberman and Tali Pinsky on a computational topology project. We studied the properties of links derived from binary quadratic forms. A link is a set of knots that may be knotted with each other as well as with themselves. A knot is an embedding of a circle in $\mathbf{R^3}$, or alternatively $\mathbf{S^3}$, which is similar to $\mathbf{R^3}$ except for the addition of a point at infinity that allows the space to 'wrap around' in all directions.

The links we studied arise as periodic orbits on the Lorenz template, which is the space occupied by the strange attractor of the Lorenz system. Figure 1 shows one possible link on the template. A link loops around the two holes on the template, and a complete description of a link is given by a set of sequences of 'hole 1' and 'hole 2' indicating which hole is to be looped around. However, care must be taken when drawing the link to ensure that one does not become 'trapped' looping around a hole without being able to complete the rest of the link. A drawing of the link enables us to encode it in a desirable way. In the first stage of the project, an algorithm was implemented in python to accomplish this task.
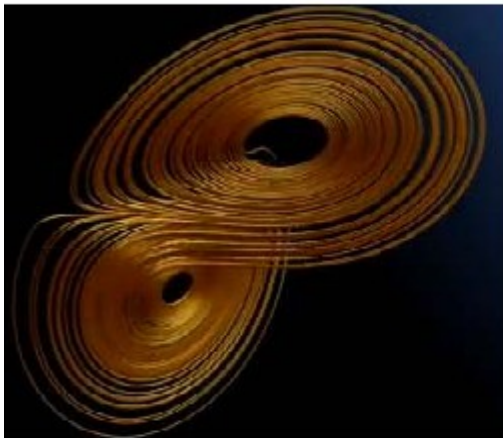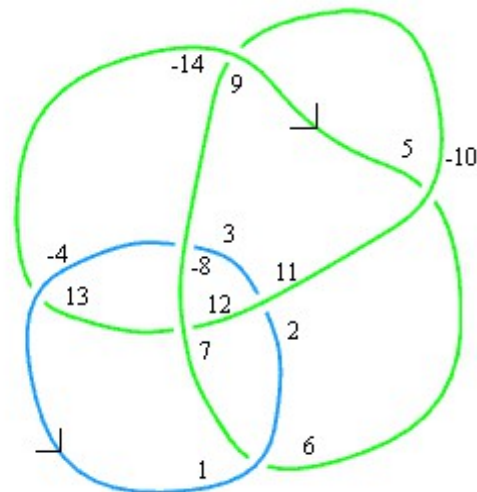

Figure 1

Figure 2

Next, in order to use snapPy software to investigate the topological properties of these links, it was necessary to encode the links in a particular way. We used Dowker-Thistlethwaite (DT) codes, which are based on the crossings of the link. A DT code is obtained by starting at an arbitrary crossing and traveling along each link component. Crossings are labeled from 1 to 2n as they are encountered, where n is the number of crossings. Each crossing is labeled with an odd number and an even number. Over-crossings with an even number are multiplied by -1. We then make a list of all the odd numbers of our labeling in increasing order and write the corresponding even numbers underneath. This list of

even numbers is the DT code. For example, the DT code for the link in Figure 2 is [(6,-8),(-10,12,14,2, -4)]. Since it is possible to modify a link in trivial ways, such as by twisting one of its strands around itself arbitrarily many times, equivalent links may have different DT codes. However, a DT code uniquely determines a link. An algorithm was developed and implemented in python to give a DT code for a link.

Given a DT code, snapPy computes a triangulation of the manifold obtained by taking the link complement in S^3, which allows us to calculate the volume of the manifold. We did this for several thousand links in hopes of finding some patterns. One of the patterns we found was that the volume of a link complement is proportional to the total length of the link measured using the hyperbolic metric. At the end of the summer, we were still interpreting the results and trying to account for scatter in the data. The next step in this project involves making conjectures about our observations and proving them rigorously.