Summer 2013 - NSERC USRA Report
Forbidden Submatrices and Configurations

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This summer I worked with Dr. Richard Anstee on the problems of forbidden submatrices and configurations, topics of extremal combinatorics. Given a $k \times l$ matrix F whose entries are all 0's or 1's (a *(0,1)-matrix*), we consider an m-rowed (0,1)-matrix A with no repeated columns (A is *simple*), and no submatrix F. We define $\text{Avoids}(m, F)$ to be the set of such matrices A, and $\text{fs}(m, F)$ to be the maximum number of columns of any $A \in \text{Avoids}(m, F)$. There is a conjecture of Anstee, Frankl, Füredi, and Pach that $\text{fs}(m, F) \in O(m^k)$. Similarly, we can consider the problem of forbidding any row or column permutation of F (a *configuration* of F), defining $\text{Avoid}(m, F)$ to be the set of simple (0,1)-matrices A with no configuration F and $\text{forb}(m, F)$ to be the maximum number of columns of such an A. We seek to prove bounds on $\text{fs}(m, F)$ and $\text{forb}(m, F)$ for specific F to gain insight.

1 Forbidden Submatrices

A structural result was achieved for the submatrix

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Keeping track of instances of $\binom{1}{0}$ and noting that if it occurs on distant rows an instance is present on every intermediate row, we motivate the following definition.

Definition 1.1 The *span* C_α of a column α is the set of rows between its top 1 and bottom 0, inclusive.

For example, if $\alpha = (0, 1, 1, 0, 1)^T$, $C_\alpha = \{2, 3, 4\}$. The following is our result.

Lemma 1.2 *There exists a matrix* $A \in Avoids(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$ *with* $|C_\alpha|$ *increasing.*

Additionally, we made some observations regarding the submatrix

$$F = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

From constructions avoiding the submatrix

$$\Gamma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

we considered columns of a given column sum and made the following definitions:

Definition 1.3 A *primary column* is a column whose top 1 is in a row where no previous column of that column sum had its top 1.

Definition 1.4 A *secondary column* is a column that is not a primary column.

Note that every secondary column creates the submatrix Γ with a primary column. There are $m - k + 1$ primary columns of B_k given by the choices of the locations of the top 1, so we aim to produce a bound on the number of secondary columns. We can assign to every secondary column β a row j such that Γ occurs with right column β and bottom row j. If row j is already associated with a row δ, we could show that a new row k could be assigned to a column to resolve this conflict. However, it is possible that a column β would be assigned a row j, creating an overlap that assigns it to row k, and conflicting with a previous column to assign it back to row j. The presence of these cycles prevented us from proving a linear bound. A number of results regarding these cycles were provided, however.

2 Forbidden Configurations

We attempted to produce a quadratic bound for the configuration

$$F = t \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using inductive techniques applied to a previous configuration, along with a result of Balogh and Bollobás, we attempted to deduce the structure of A.

References

[1] R.P. Anstee and Linyuan Lu, Repeated Columns and an old chestnut, submitted to *Elec. J. of Combinatorics*

[2] R.P. Anstee, A Survey of forbidden configurations results, *Elec. J. of Combinatorics* **20** (2013), DS20, 56pp.