FREE FINITE GROUP ACTIONS ON RIEMANN SURFACES

CHRIS SMITH

This summer I had the privilege and pleasure of working with Dr. Alejandro Adem in the field of algebraic topology with the goal of classifying certain free group actions on surfaces of minimal genus. The bulk of the research dealt with the topic of compact, orientable surfaces (henceforth referred to as just "surfaces"), and classifying to the greatest extent possible those groups which act on them. The definition of "minimal" used here will be explained after some further background information.

We begin by using a well-understood equation involving the Euler characteristic of triangulations of a given surface. Given a surface M, the Euler characteristic is defined as $\chi(X) = f - e + v$ where f, e, v are the number of faces, edges and vertices in a single given triangulation of the surface. As the characteristic is invariant with respect to the choice of triangulation, this is well defined. For orientable surfaces, we obtain $\chi(M_g) = 2 - 2g$ while for non-orientable we have $\chi(M_g) = 2 - g$ where g is the genus of the surface.

Using lifting theorems, it can be shown that an n-sheeted covering space $\tilde{X}$ of X will necessarily have an Euler characteristic satisfying $\chi(\tilde{X}) = n \cdot \chi(X)$. As the surface of genus 0 is merely the sphere, and the surface of genus 1 gives the equation $0 = 0n$ making it a somewhat special case, we interest ourselves here in the situation where $X = M_2$ is the (orientable) surface of genus 2. Now supposing we have a free action of some finite group G on a surface of genus g, we know that this gives rise to a quotient space M_g/G for which the surface M_g is a $|G|$-sheeted cover, and hence we obtain the equation $\chi(M_g) = |G|\chi(M_g/G)$, or $2 - 2g = |G|(2 - 2g')$ where g' is the genus of M_g/G. This tells us that G can only possibly act on a surface M_g if $|G|$ divides $1 - g$.

The fundamental group of M_g is:

$$\pi_1(M_g) = F_{2g}/\langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle_N$$

We understand that every subgroup of a fundamental group is the fundamental group of a covering space, and similarly every covering space has a fundamental group isomorphic to the subgroup of the fundamental group of the space it covers. Furthermore, we understand that G acts freely on M_g if and only if there exists an exact sequence $1 \to \pi_1(M_g) \to \pi_1(M_{g'}) \to G \to 1$ for some surface $M_{g'}$, and that this is the case if and only if there exists a surjective homomorphism $\varphi : \pi_1(M_{g'}) \to G$. Clearly if this is true for g' then it will hold for $g' + 1$ (use the homomorphism obtained for g', and map the elements $a_{g'+1}$ and $b_{g'+1}$ to the identity to obtain a new surjective homomorphism $\pi_1(M_{g'+1}) \to G$). This, then, is how we finally define minimality. As we are disinterested in the cases $g' = 0, 1$ we are looking for the existence of surjections $\pi_1(M_2) \to G$, or in other words, the existence of a free action of G on the surface of genus $1 + |G|$. If G acts freely on $M_{1+|G|}$, then it acts freely on all surfaces satisfying the Euler characteristic equation above for $g \geq 2$. The "surface of minimal genus" for a given group is hence defined to be the surface $M_{1+|G|}$.

This greatly reduces the problem, as several large categories of groups can be shown to satisfy these conditions relatively trivially. For example, if a group $G = \langle g, h \rangle$ is of rank at most

2 (where rank here refers to the cardinality of a minimal generating set of the group), then we can construct the surjective homomorphism by mapping $a_1 \mapsto g$, $a_2 \mapsto h$, $b_1 \mapsto e$, $b_2 \mapsto e$ (where e is the identity, and a_i, b_i are the generators of $\pi(M_g)$). We can make a similar map for cases where G is the direct product of two rank 2 groups (mapping the elements b_i to the second group in the product). This solves the problem for a large number of groups, including many of the commonly used ones such as:

- The dihedral groups
- The symmetric groups
- The alternating groups
- All simple groups
- The direct product of any two of the above

For abelian groups an even weaker condition is sufficient: abelian groups of rank up to 4 all act freely on $M_{1+|G|}$. This is clear because the relators of $\pi_1(M_g)$ are a subset of its commutator subgroup. In fact, any group generated by three elements of which two commute, or generated by four elements which commute in pairs (that is, $G = \langle a, b, c, d \rangle$ with $[a, b] = [c, d] = e$) act freely on $M_{1+|G|}$. This, for example, gives us all groups of the form $G = N \rtimes K$ with N rank 2 abelian and K cyclic.

The question quickly becomes, which groups do *not* act freely on the surface of minimal genus? Proving that a group does not act freely on a surface of minimal genus proves to be more challenging than showing that it does act freely. While it can occasionally be done easily (as in the case where the group is of rank greater than 4), for the most part this resulted in computational methods taking every set of four (not necessarily distinct) elements which could generate the group, and testing if there existed a well defined homomorphism mapping the generators of $\pi(M_g)$ to those elements. With the available time and computational power, complete results were computed for groups of order up to 128.

Results. The smallest abelian group failing to act freely was of course $(\mathbb{Z}/2\mathbb{Z})^5$, and the smallest non-abelian group was $(\mathbb{Z}/3\mathbb{Z})^3 \rtimes (\mathbb{Z}/2\mathbb{Z})$ with $\mathbb{Z}/2\mathbb{Z}$ acting on each generator of the normal subgroup individually and non-trivially. Approximately 97.95% of groups of order at most 127 act freely, and 94.66% of groups of order at most 128 do, in comparison to 96.59% of groups of order at most 64.

REFERENCES

[1] Allan Clark, *Elements of Abstract Algebra*, 1971.
[2] A. Howard M. Hoare, Abraham Karrass, and Donald Solitar, *Subgroups of Infinite Index in Fuchsian Groups*, Mathematische Zeitschrift **125** (1972), 59–69.
[3] W. S. Massey, *Algebraic Topology: An Introduction*, 1967.
[4] GA Miller, *Possible Orders of Two Generators of the Alternating and of the Symmetric Group*, Transactions of the American Mathematical Society **30** (1928), 24–32.
[5] Aner Shalev, *Asymptotic Group Theory*, Notices of the AMS **48** (2001), 383–389.