The effects of spatial dimensions in 2x2 games
NSERC USRA Report

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1 Introduction

Evolutionary game theory is a way to study biological interactions, and gain a greater understanding of the outcomes. It has been shown that adding 2 dimensional spatial structure can promote cooperation in certain games, where a well mixed population, one where every individual interacts with the whole population, cooperation cannot survive. However, many organisms live in 3 dimensions, rather than 2, and so investigating the evolution of cooperation in 3 dimensions is important. In my report I will briefly describe the effect of 3 dimensional and 4 dimensional structure on 2x2 games.

Here is a 2x2 payoff matrix which describes the payoffs of Player 1, given Player 1 and Player 2's choices

$$
\begin{array}{cc}
 & \text{Player 2} \\
 & \begin{array}{cc} C & D \end{array} \\
\text{Player 1} \quad \begin{array}{c} C \\ D \end{array} & \begin{array}{|c|c|} \hline R & S \\ \hline T & P \\ \hline \end{array}
\end{array}
$$

where the way in which R, S, T, P are ordered determines the game which is being played. The general payoff matrix has four parameters, and without loss of generality we can normalize it so that $R = 1$ and $P = 0$ holds, and then we are left with two parameters, S and T. When we normalize we are assuming that $R > P$, if this is not true then we simply switch C and D so it holds. We can then create a two-dimensional parameter space with S and T to describe various different games. Here C refers to the strategy of 'cooperation', and D refers to 'defection'. Therefore players that choose C will be refered to as 'cooperators' and players that choose D will be referred to as defectors.

To look at the effect of spatial structure on cooperation, we generate a lattice with 125000 individuals, so that each individual interacts with the individuals directly connected to them. In 2 dimensions, this is a square lattice, and in 3 dimensions it is a cubic lattice. The individuals then interact for 1000 generations of updating, where an individual i changed to individual j's strategy with probability $P_{i \to j} = \frac{1}{1+e^{\delta(\pi_i - \pi_j)}}$, where δ represents the strength of selection and π_i and π_j represent the payoffs of individuals i and j respectively. The system was the averaged for 100 generations, and the equilibrium fraction of cooperators was observed. Figure 1 shows the equilibrium fraction of cooperators as a function of the parameters S, T. S, T were taken from the interval $[-5, 5]$, and incremented in steps of 0.2.

2 Results

In Figure 1 we can see that for the quadrant $S < 0, T < 1$ the cubic lattice promotes cooperation over a well-mixed population, but the square lattice promotes cooperation more than the cubic lattice. Qualitatively, for most of the S, T plane, the equilibrium fraction of cooperators on the 3D structure is a mix between that of the 2D structure and the well mixed population. However, when we look closely at the Prisoner's Dilemma game we find something surprising.

Here we reduce the payoff matrix to one parameter r so we are now looking at a transect through
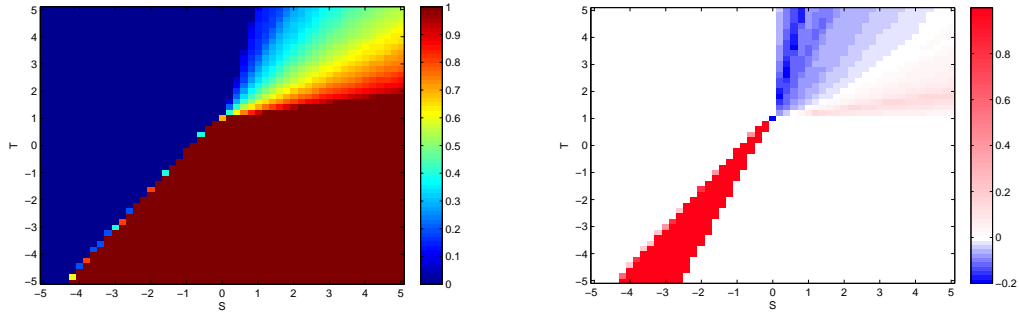
(a) Equilibrium fraction of cooperators on the 3D lattice in the S, T plane

(b) Difference in equilibrium fraction of cooperators between 3D and well mixed population
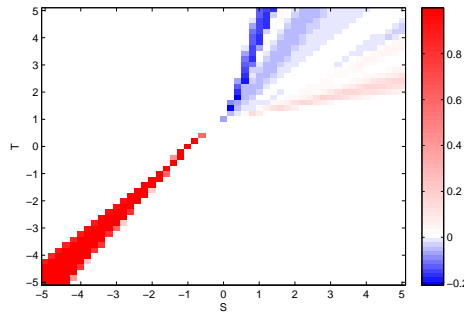
(c) Difference in equilibrium fraction of cooperators between 2D and 3D

Figure 1: In (a) red represents a high equilibrium fraction of cooperators, and blue represents a low fraction. In (b) red indicates that the cubic lattice promotes cooperation more than the mean field, and blue means that the cubic lattice inhibits cooperation over the mean field. In (c) red indicates high cooperation on the square lattice over the cubic lattice, and blue indicates lower cooperation.
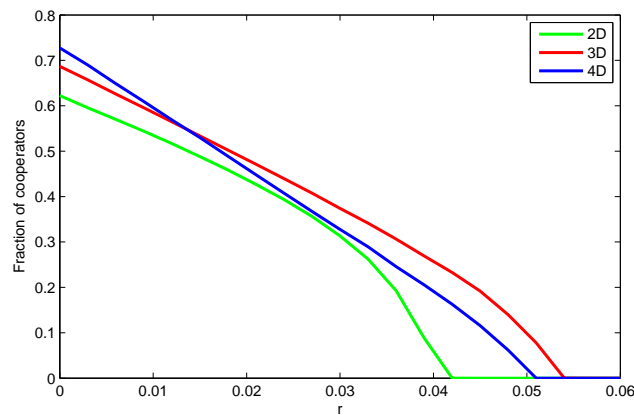

Figure 2: Here we can see the equilibrium fraction of cooperators as a function of r in the Prisoner's Dilemma Game. For each value of r the simulations were run for 10000 generations, and then another 1000 generations were averaged. The rest of the specifications were the same as Figure 1.

the Prisoner's Dilemma Game.

$$
\begin{array}{c|c|c|}
 & C & D \\
\hline
C & 1 & S \\
\hline
D & T & 0 \\
\hline
\end{array}
\Rightarrow
\begin{array}{c|c|c|}
 & C & D \\
\hline
C & 1 & -r \\
\hline
D & 1+r & 0 \\
\hline
\end{array}
$$

We decided to also look at the 4 dimensional lattice, and the equilibrium fraction of cooperators are shown for different values of r in Figure 2. The simulations were run to the same specifications as in Figure 1.

In the Prisoner's Dilemma, in the well mixed population, cooperation cannot survive for any value of r. Here we find that 3D and even 4D spatial structure promote cooperation more than 2D for all values of r where cooperation survives.

I would like to thank Dr. Hauert for supervising me on this project, and I hope to definitely pursue this subject further in the future.

3   References

Hauert, Ch. (2002) Effects of space in 2x2 games, Int. J. Bifurcation Chaos 12, 1531-1548.