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PERCOLATION AND CONTACT PROCESS THROUGH INVASION METHODS

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Take the graph formed by setting your vertices as $\mathbb{Z}^d$ and including an edge between two vertices x and y iff $0 < \|x-y\|_\infty \leq R$, where R is some positive integer parameter. Then, take the sub-graph formed by including all vertices and including each edge with probability p. It is known that there exists $p_c(d,R) \in [0,1]$ such that if $p > p_c$, there exists an infinite connected sub-graph of this sub-graph with probability 1 and that if $p < p_c$, there exists an infinite connected sub-graph of this sub-graph with probability 0. The purpose of this project is to numerically investigate the asymptotic behaviour of p_c for $d = 2, 3$ at high R.

In order to get approximations of the value of p_c, we use the method of invasion percolation. Take the original graph (before randomly selecting edges) and assign a weighting to each edge from 0 to 1 according to a uniform random variable. Pick any vertex and consider the set containing only it to be G_0. Define S_{i+1} as the lowest weight of an edge connecting an element of G_i to an element g_{i+1} of G_i^C and define G_{i+1} as $G_i \cup \{g_{i+1}\}$. Then, it is known that $\limsup_n S_n = p_c$.

Using a C++ program written to simulate this process and analyzing the results in Octave, I was able to numerically confirm the conjectures [1] that for large R, $p_c V(R) \approx 1 + \frac{\theta_2}{R}$ in $d = 2$ and that $p_c V(R) \approx 1 + \frac{\theta_3}{R^2}$ in $d = 3$, where θ_2, θ_3 are constants and $V(R) = (2R+1)^d - 1$ is the connectivity of the original graph. We also found that $\theta_2 \approx 0.7$ and $\theta_3 \approx 1.2$, however, due to the unknown rate of convergence of the lim sups used, we cannot give accurate error ranges on these values.

In addition, we worked on the contact process problem. Consider the set $\mathbb{Z}$. Some finite subset I of $\mathbb{Z}$ is considered infected as initial conditions and the remainder I^C is considered uninfected. Each element of I will recover from infection, and become no longer an element of I after a time determined by an exponential random variable with parameter 1. Any uninfected element x will become infected with an instantaneous infection rate at time t of $q(x,t)\lambda$, where $q(x,t) = \frac{|\{|y-x| \leq R|(y,t) \in I\}|}{2R}$, or the fraction of points within a distance of R from x that are infected at time t. It is known that there exists a $\lambda_c > 0$ such that for $\lambda > \lambda_c$, the disease will survive indefinitely with nonzero probability, while for $\lambda < \lambda_c$, it will die out after a finite amount of time with probability 1 [2].

We conjectured an analogous method to invasion percolation for the contact process. Make random connections between integers at random times with even densities everywhere (Poisson variable for the number of connections between any two integers over any time interval of length δt with mean $\frac{\lambda_{max}\delta t}{2R}$), where $\lambda_{max} >> \lambda_c$ and weight each of the connections with a random number uniformly distributed between 0 and λ_{max}. Next, start with some finite infected set I and for each element of it,determine when it will be uninfected (a random exp(1) variable). Follow the lowest value connection from an infected point in space-time to an uninfected one and add the interval starting with that point (and lasting an exp(1) duration or until it was already infected, whichever is shorter) to the infected set. Unless the process terminates early (no connections exist between infected and uninfected points), we conjecture that the sequence of weightings of connections followed, S, will have the property that $\limsup S = \lambda_c$. In addition, for a sufficiently large λ_{max}, the probability of an early termination can be made arbitrarily small.

Using another C++ program written to simulate this, we confirmed the conjecture that for large R, $\lambda_c \approx 1 + \frac{\theta}{R^{\frac{2}{3}}}$ [3] and pinned down that $\theta_c \approx 2.1$, although, similarly to invasion percolation, exact error ranges could not be determined.

Works Cited

[1] Perkins, Edwin. Personal communication.
[2] Liggett, Tom. *Interacting Particle Systems*. New York: Springer-Verlag, 1985. 300-02.
[3] Durrett, Richard, and Perkins, Edwin. "Rescaled Contact Processes Converge to Super-Brownian Motion for d greater than or equal to 2." *Prob. Theory and Related Fields* 114 (1999): 309-99.