NSERC USRA Report: The Influence of Delay in Gene Expression Models and Their Associated Nonlocal Eigenvalue Problems

Nabil Fadai

Summer 2013

1 Introduction

In a paper written by Seirin Lee et al. (2010), the influence of gene expression time delays on Gierer-Meinhardt pattern formation systems is discussed. The original model being modified stems from a Turing model for the interaction of two morphogens (an activator and an inhibitor) and their kinetics over long periods of time. Seirin Lee et al. show that spatial instabilities in both morphogens occur when the time delay is relatively large. However, their linear analysis of the system predicted that time delays would not influence the stability of the system, to which they suggest that additional analysis is required.

In this project, I further investigated the stability of the Gierer-Meinhardt system studied by Seirin Lee et al., using linearizations about a single quasi-equilibrium "pulse" solution as opposed to constant solutions. By examining the stability of the system, we are left with equations that relate the eigenvalues and eigenfunctions of the system together, known as non-local eigenvalue problems (NLEPs). In some systems, such as the one studied by Seirin Lee et al., the NLEP is explicitly solvable, and a region of stability can be constructed to determine exactly where one-pulse solutions lose their stability, in terms of additional parameters of the system. This in turn led to the discussion of the stability of more general one-dimensional Gierer-Meinhardt systems and their associated NLEPs, which may or may not be explicitly solvable. Finally, the stability of two-dimensional Gierer-Meinhardt systems and their associated NLEPs was discussed, where numerical methods of computing eigenfunctions were required. In all systems discussed, the behaviour of eigenvalues as $T \to \infty$ is also discussed.

2 Results

For the system studied by Seirin Lee et al., i.e.

$$\begin{cases} v_t = \epsilon^2 v_{xx} - v + \frac{v^3}{u_T^2}, & x \in \mathbb{R}, \quad t > 0, \quad u_T = u(t-T, x) \\ \tau u_t = D u_{xx} - u + \frac{1}{\epsilon} v^3, & \lim_{x \to \pm\infty} u(t,x) = \lim_{x \to \pm\infty} v(t,x) = 0 \end{cases} \tag{2.1}$$

where T is the time delay, it was determined that for $T > \frac{1}{\sqrt{72}} \arctan(2\sqrt{2}) \approx 0.14507$, one-pulse equilibrium solutions (and hence, N-pulse solutions) are always unstable for all τ. The stability diagram for this system is shown in Figure 2.1. Additionally, it was determined that there exists infinite number of Hopf bifurcations in the system, showing that there are infinitely many unstable eigenvalues as $T \to \infty$. However, all of these eigenvalues except one approach the origin; the paths of these eigenvalues are shown in Figure 2.2. In the limiting case when there is fast diffusion, i.e. $D \to \infty$, the region of stability shrinks in height in the (T, τ) plane. As $T, D \to \infty$, the two-term asymptotic expansion of the eigenvalues is

$$\lambda = \frac{\ln 3}{T} + \frac{2n\pi i}{T} + \left(\frac{\ln 3}{T^2} + \frac{2n\pi i}{T^2} \right) \left(\frac{1}{3} - \tau \right) + O\left(\frac{1}{T^3} \right)$$

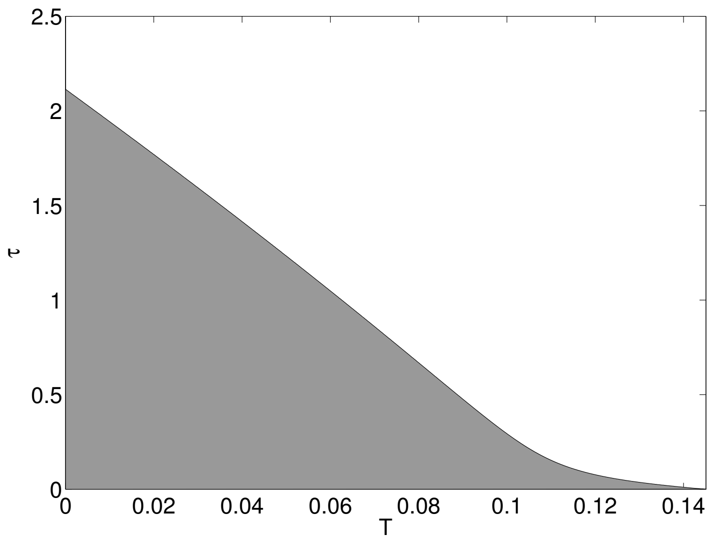

Figure 2.1: The stability region in the (T, τ) plane for system (2.1).

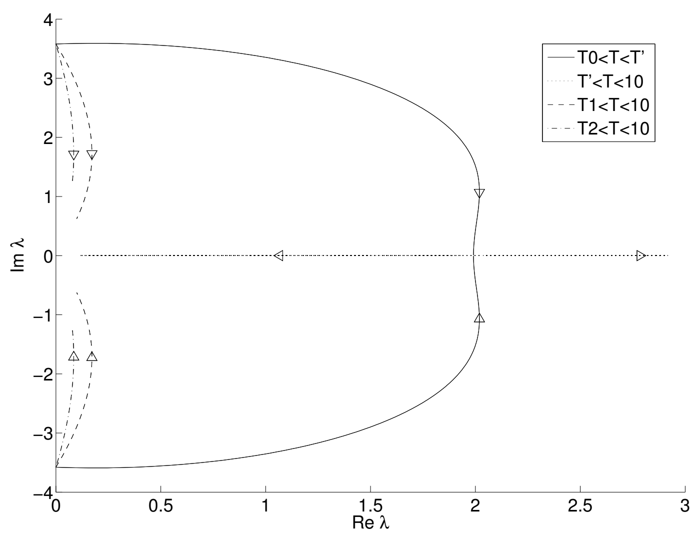

Figure 2.2: Paths of unstable eigenvalues for $T \in [T_0, 10]$, with $\tau = 1$. Note that T' corresponds to when the first real eigenvalue appears, and $T_1,\ T_2$ correspond to when additional Hopf bifurcations occur. Here, $T_0 \approx 0.0626,\ T' \approx 0.8239,\ T_1 \approx 1.8186$, and $T_2 \approx 3.5746$.

In discussing the stability of an N-pulse solution in the two-dimensional Gierer-Meinhardt system

$$\begin{cases} v_t = \epsilon^2 \Delta v - v + \frac{v^2}{u_T}, & (x,y) \in \Omega, \quad t > 0, \quad u_T = u(t-T, x) \\ \tau u_t = D\Delta u - u + \frac{1}{\epsilon^2} v^2, & u_n,\ v_n = 0,\ (x,y) \in \partial\Omega \end{cases} \tag{2.2}$$

the NLEP problem is not explicitly solvable and therefore an exact stability region cannot be determined. However, an appoximate stability region can be constructed in terms of T, τ and dimensionless parameter

$\mu = \frac{2\pi N D_0}{|\Omega|}$, where $|\Omega|$ is the area of the domain, $D_0 = D\nu$ and $\nu = -\frac{1}{\ln \epsilon}$. In the case where the pulses are synchronous, the boundary of the stability can be described as a surface, shown in Figure 2.3. The two-term asymptotic expansion of eigenvalues for large delay in this model is not well understood, as it relies on an adjoint eigenfunction problem that is currently unsolved.

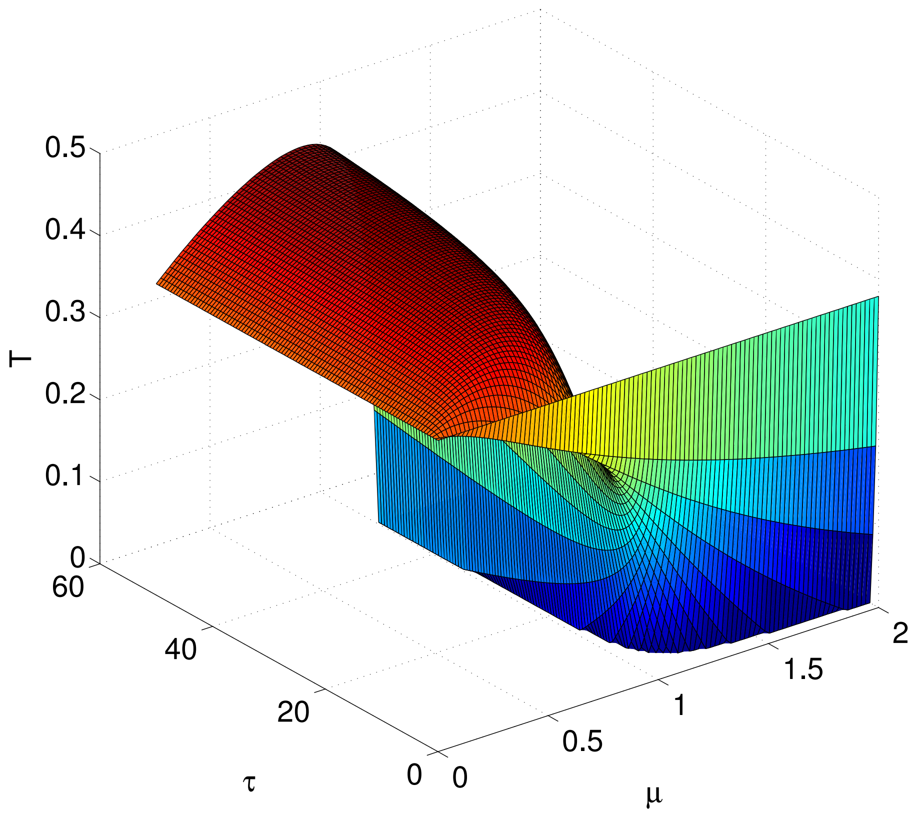

Figure 2.3: The approximate boundary of stability for the synchronous mode of (2.2). Any (μ, τ, T) triple that lies below the surface is stable.

3 References/Acknowledgements

Seirin Lee, S., Gaffney, E. A., & Monk, N. A. M. (2010). The Influence of Gene Expression Time Delays on Gierer–Meinhardt Pattern Formation Systems. Bulletin of Mathematical Biology 72: 2139-2160. Springer-Verlag.

Special thanks to Dr. Michael Ward for his supervision on this project.