Summer 2012 — NSERC USRA Report
Forbidden Submatrices

Ron Estrin

Over the course of the summer, I worked with Dr. Richard Anstee on problems concerning forbidden submatrices, a topic of extremal set theory. We work with matrices that contain entries which are 0 or 1, and are thus called *(0,1)-matrices*, and ask the following question: given a $k \times \ell$ matrix F, what is the maximum number of columns (without repetition) that an m-rowed matrix A can have, such that it does not contain F as a submatrix? The overall goal behind this research is to prove the following conjecture:

Conjecture. *[AZ86]Let F be a $k \times \ell$ (0,1)-matrix. Then there exists some constant c_F such that for any $m \times n$ (0,1)-matrix A with no repeated columns and no submatrix F,*

$$n \leq c_F m^k.$$

We define $fs(m, F)$ as the maximum number of columns that an m-rowed (0,1)-matrix with no repeating columns can have while forbidding F.

Bounds for Arbitrary F

Although these problems are closely related to forbidden configurations, where we are forbidding any row and column permutation of F in A, forbidden submatrices require that the row and column order be preserved. But there are often times when theorems concerning configurations are useful in proving bounds on the number of columns in forbidden submatrix problems. We used configurations in the process of proving bounds on arbitrary F. The first bound is an improvement of previous work in [Ans00]:

Theorem. *Let F be a $k \times \ell$ (0,1)-matrix. Then $fs(m, F) \leq c_F m^{2k-1-\epsilon}$, where $\epsilon = (k-1)/(6\log_2 \ell)$ for some c_F.*

Theorem. *Let F be a $k \times \ell$ (0,1)-matrix. Then fs(m,F) is $fs(m, F) \leq c_F m^{2k-1-\epsilon}$, where $\epsilon = k(k-1)/(13\log_2\log_2 \ell + \log_2 \ell)$ for some c_F.*

Theorem. *Let F be a $k \times \ell$ (0,1)-matrix. Then fs(m,F) is $fs(m, F) \leq c_F m^{2k-1-\epsilon}$, where $\epsilon = k(k-1)/(10k + \log_2 \ell)$ for some c_F.*

Although these bounds are for any F, they are far from proving the conjecture.

Decomposition of a Digraph

The majority of our work was on a new way of decomposing directed graphs. We use directed graphs to describe the current state of a matrix A, as we search through it for whether it contains F. The idea is that every vertex represents a row in the next possible column to appear (thus taking on a value of 0 or 1), and have edges between them depending on what values begin to make a copy of F on those rows.

By looking for certain patterns in the possible digraphs, we can force values onto some vertices, and discover relationships between the values of groups of vertices. For example, structures may arise where a set of vertices must always have the same value, while some other set of vertices must contain the opposite set of values, to avoid creating F. Using such structures to decompose a digraph, although we were not able to prove new bounds, we were able to find new proofs for previous results, some with improved bounds.

Theorem. *[AC11]Let F be the $2 \times \ell$ matrix*

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & ... \\ 0 & 1 & 0 & 1 & 0 & ... \end{bmatrix}.$$

Then for any $m \times n$ (0,1)-matrix with no repeated columns and no submatrix F,

$$n \leq 2(\ell - 1)\binom{m}{2} + m + 1$$

Theorem. *[AC11]Let F be the $2 \times \ell$ matrix*

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & ... \\ 1 & 0 & 1 & 0 & 1 & ... \end{bmatrix}.$$

Then for any $m \times n$ (0,1)-matrix with no repeated columns and no submatrix F,

$$n \leq 5(\ell - 1)\binom{m}{2} + 2m + 2$$

Forbidding a Family of Columns

One other problem that we worked on this summer was the how many (0,1)-columns exist that avoid some set of forbidden columns. This was an interesting side problem to consider, and resulted in an interesting asymptotic bound.

Theorem. *Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$be a set of (0,1)-columns of length $k_1, k_2, ..., k_n$ respectively. Then*

$$c_1 m^{t-1} \leq fs(m, S) \leq c_2 m^{t-1}$$

where t is the length of a longest common subsequence of the columns in S.

References

[AC11] R.P. Anstee and R. Chen, *Forbidden submatrices: some new bounds and constructions*, submitted to Elec. J. of Combin. (2011), 13pp.

[Ans00] R.P. Anstee, *On a conjecture concerning forbidden submatrices*, J. Combin. Math. and Combin. Comp. **32** (2000), 185–192.

[AZ86] R.P. Anstee and Füredi Z, *Forbidden submatrices*, Discrete Mathematics **62** (1986), 225–243.