Problems in density Ramsey theory

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This summer we worked with Professor Akos Magyar and Professor Malabika Pramanik on problems in additive combinatorics/Ramsey theory, a field which seeks to establish the existence of ordered substructures within objects of minimal structure themselves, such as arbitrary subsets of the integers. Historical approaches to problems in the field have included combinatorial, ergodic, and Fourier-analytic arguments. In our project, we learned about some of the common Fourier-analytic techniques and sought to apply them to some new problems.

Central to our project was the machinery used in establishing Roth's theorem, later extended by Gowers [1] to prove the stronger theorem of Szemerédi. The latter states, among other equivalent formulations, that for any $\delta > 0$ and positive integer k, there will exist some positive integer N such that any subset of $[1, N] \cap \mathbb{Z}$ of cardinality at least δN will contain a proper arithmetic progression of length k. Roth's theorem is the special case of this with $k = 3$.

Our main problem was to establish results in a similar vein over finite fields, say $Z = \mathbb{F}_p^n$, but with geometric squares instead of arithmetic progressions. Here, a square is a set of four points $\{x, x+u, x+v, x+u+v\}$ such that $u \cdot v = 0$ and $|u|^2 = |v|^2$, with the standard inner product when treating Z as a vector space over $\mathbb{F}_p$. In the process, we also looked at the analogous problems for isoceles triangles, right triangles, and isoceles right triangles, defined similarly. Here we use the concept of uniformity, which was important in the proof of Roth's theorem. The linear bias $\|A\|_u$ of a set $A \subset Z$ is defined to be

$$\|A\|_u = \max_{\xi \in Z \setminus \{0\}} |\hat{1}_A(\xi)|,$$

where $\hat{1}_A$ is the Fourier transform of 1_A, the indicator function of A. We shall refer to a set $A \subset Z$ as uniform if it has low linear bias. In what follows, take δ to be the function such that $\delta(0) = 1$ and $\delta(t) = 0$ otherwise, and for notational convenience, define $\tilde{1}_A(x, u, v) = 1_A(x)1_A(x+u)1_A(x+u+v)$ and $\mathbb{P}(A) = |A|/|Z|$. The results were:

Proposition. *For a uniform set $A \subset Z$, A contains close to the expected number of isoceles* *triangles, right triangles, and isoceles right triangles. Quantitatively, for right triangles:*

$$\left|\mathbb{E}_{x,u,v\in Z}\tilde{1}_A(x,u,v)\delta(u\cdot v) - \frac{\mathbb{P}(A)^3}{p}\right| \le \mathbb{P}(A)^2\left(\frac{1}{p^n} + \|A\|_u\right).$$

For isoceles triangles:

$$\left|\mathbb{E}_{x,u,v\in Z}\tilde{1}_A(x,u,v)\delta(|u|^2-|v|^2) - \frac{\mathbb{P}(A)^3}{p}\right| \le \frac{\mathbb{P}(A)^2}{p^n} + \mathbb{P}(A)\|A\|_u.$$

For isoceles right triangles:

$$\left|\mathbb{E}_{x,u,v\in Z}\tilde{1}_A(x,u,v)\delta(u\cdot v)\delta(|u|^2-|v|^2) - \frac{\mathbb{P}(A)^3}{p}\right| \le \frac{\mathbb{P}(A)^3}{p^{n/2}} + \frac{\mathbb{P}(A)^2}{p^n} + \left(1+p^{-n/2}\right)\mathbb{P}(A)\|A\|_u.$$

Here, we say "expected" as based on naively treating the events of x, $x+u$, and $x+u+v$ falling in A as probabilistically independent. Each constraint further introduces a factor of $1/p$, if we assume that the expressions $u\cdot v$ and $|u|^2-|v|^2$ are uniformly distributed among the p possible values. With regards to squares, we have the following result for rectangles:

Proposition. *A set $A\subset Z$ will contain many rectangles:*

$$\left|\mathbb{E}_{x,u,v\in Z}\tilde{1}_A(x,u,v)1_A(x+u+v)\delta(u\cdot v)\right| \ge \frac{\mathbb{P}(A)^4}{p}$$

To arrive at an estimate for squares, we treated the rectangle estimate as the main contributing term, and bounded the error term (counting rectangles that are not squares) from above. This was done in two separate ways, one of which did not require the assumption of uniformity. In any case, via a intermediate result from the proof of Roth's theorem, the assumption of uniformity can be dropped.

The specific case of squares on the two-dimensional space $\mathbb{F}_p^2$ was also examined. It was established that the previous criterion of uniformity was insufficient via an explicit probabilistic construction of uniform sets in $\mathbb{F}_p^2$ that have fewer than the expected number of squares, adapted from a construction of Gowers. However, we found that the number of squares can be controlled by the U^3 Gowers norm, and this was established in a proof closely emulating that of Gowers [1][2] for the $k=4$ case in Szemerédi's theorem, which also utilized the U^3 norm.

References

[1] W. T. Gowers. A new proof of Szemerédi's theorem. *Geometric and Functional Analysis.*

[2] T. Tao and V. Vu. *Additive combinatorics.*