# Summer 2012
NSERC USRA Report
Random Quadrangulations
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This summer I had the privilege and pleasure of working with Professors Omer Angel and Lior Silberman, and fellow students Edwin Chen and Yuqi Zhu on the topic of random graphs, in particular, random quadrangulations. A planar map is called a quadrangulation if every face has four sides on its boundary. For example,
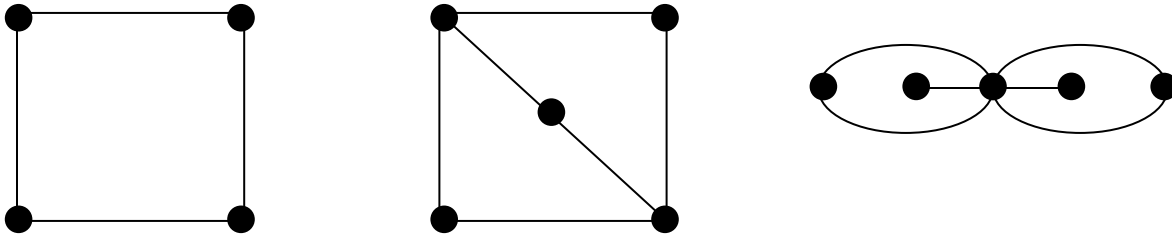
are three examples of quadrangulations. The "outside" face must also have four sides. Also, as illustrated in the third example, the edge that lies entirely inside a face must be counted twice.

Generation of Random Quadrangulations

There are various ways of generating random quadrangulations of fixed size uniformly (meaning that each quadrangulation gets equal chance of being generated). My main duty was to write computer programs to generate these random structures, and provide professors with numerical evidence. One way to generate a quadrangulation is to generate a uniform 4-regular planar graph, and take the dual graph. Another way is to generate a uniform well-labelled planar tree, and construct the quadrangulation via what is known as Schaeffer's bijection. The latter approach is computationally more efficient, and Python script was written to implement this method. In fact, using an infinite tree, Schaeffer's bijection allows one to define Uniform Infinite Planar Quadrangulation (henceforth called UIPQ). Since finite computer memory cannot store such object, we resort to experimenting with very large but finite quadrangulations, which allows us to predict limiting behaviour for UIPQ.

Spectrum of Random Quadrangulations

One can look at adjacency matrix (or operator) of random quadrangulations (or any graph in general) to study its eigenvalues. It turns out that some local structures inside the graph causes certain eigenvalues to appear in the spectrum. Using computer simulation, we spent some time studying these localized features. Inspired by these observations, we were able to prove that a

dense subset of the interval $(-1, 1)$ will appear in the spectrum of quadrangulation with positive probability. We also looked at the multiplicity of zero eigenvalue for a random quadrangulation of size n. Computer simulation showed that about one quarter of the eigenvalues are zero, and the proportion getting closer to 1/4 as n increases. Although we weren't able to prove results about the exact multiplicity of zero eigenvalues, this question led us to a procedure of removing certain vertices in quadrangulation, and studying the resulting simplified graph.

Computing probabilities on UIPQ

Purely graph-theoretic properties of quadrangulations are also interesting. It can be shown that in UIPQ, the probability that a vertex has degree 1 is exactly 1/3. A vertex with degree one is called a "leaf". And a vertex that is adjacent to a "leaf" will be called a "parent". We proved that in UIPQ, the probability that a vertex is a parent is given by

$$\frac{4}{9 + \sqrt{57}}$$

This probability on its own might not be very exciting, but the method we used to calculate this value seems rather novel, and can be applied to solving similar problems on this setting. The method, which involves recursively traversing the tree, boils down to analyzing the following recurrence relation:

$$3d_n = (1 - d_n)(d_{n-1} + d_n + d_{n+1})$$

which we solved under the conditions that $0 < d_n < 1$, and $d_n \to 0$ as $n \to \infty$.

Conjecture on Subcriticality

Finally, we investigated some percolation-theoretic problems. One can ask what happens to the remaining graph when we remove both "leaves" and "parents" from the quadrangulation. We conjectured that, if this procedure is done on UIPQ, no infinitely large connected component will be left. Computer simulation gives somewhat compelling reason to believe why this conjecture might be true: By removing "leafs" and "parents" in a finite quadrangulation, we observed that the size of the largest connected component grows very (logarithmically) slowly.

Besides research, I found NSERC USRA to be particularly rewarding experience in terms of learning. Most of these topics are specialized, and not covered in an undergraduate course. I was fortunate enough to be introduced to some of these beautiful branches of mathematics. Some of these subject areas are so fascinating (especially, percolation and scaling limits) that I will make sure to explore them in more depth as a part of my future studies.