Analysis Qualifying Exam

University of British Columbia

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1. (a) Let $h : [-\pi, \pi] \to \mathbb{R}$ be defined by $h(x) = |x|$. Compute the Fourier series of h. Note — not the Fourier transform.

   (b) Let $E = \{v \in \mathbb{R}^3 \text{ s.t. } 0 < |v| \leq 1\}$ be the unit ball centred at the origin, but not including the origin. Let

$$\mathbf{F}(x, y, z) = \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right).$$

   Prove that $\operatorname{div} \mathbf{F} = 0$ on E.

   (c) Continuing from (b), show that there is no vector function $\mathbf{G} = \mathbf{G}(x, y, z)$ which satisfies $\operatorname{curl} \mathbf{G} = \mathbf{F}$ in the domain E.

   Hint: Split the surface and apply Stokes' Theorem.

2. Your answers to this question should probably contain ε's.

   (a) Prove the following (the Weierstrass M-test):
   Let A be a set, and suppose that $\{f_n\}$ is a sequence of functions from $A \to \mathbb{R}$. Further let $\{M_n\}$ be a sequence of positive numbers so that

$$\sum_{n=1}^{\infty} M_n < \infty \qquad \text{and} \qquad |f_n(x)| < M_n \text{ for all } x \in A$$

   then the series $\sum_{n=1}^{\infty} f(x)$ converges absolutely and uniformly for all $x \in A$.

   (b) Suppose that

$$g(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}.$$

   Evaluate the following:

$$I = \int_0^{\pi/2} g(x)\, dx.$$

   Your answer should take the form of an infinite series of rationals. Prove your answer.

3. Your answers to this question should probably contain ε's.

   (a) Let $g : [a, b] \to \mathbb{R}$. Carefully define what it means for g to be Riemann integrable on $[a, b]$.

(b) Prove the following from the definitions of infimum and supremum:
Let $f : [a, b] \to \mathbb{R}$. Let

$$M = \sup \{f(x) \text{ s.t. } x \in [a, b]\} \qquad\qquad m = \inf \{f(x) \text{ s.t. } x \in [a, b]\}.$$

Then

$$M - m = \sup \{|f(x) - f(y)| \text{ s.t. } x, y \in [a, b]\}.$$

(c) Prove the following from first principles:
If f is Riemann integrable on $[a, b]$, then $|f|$ is also Riemann integrable on $[a, b]$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let

$$\varphi = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}i, \qquad\qquad \text{where } i = \sqrt{-1}.$$

   (a) If f is harmonic, show that φ is holomorphic.

   (b) If φ is holomorphic, show that f is harmonic.

5. Let A and α be positive constants with $\alpha < 1$. Let f be an entire function (holomorphic on all of $\mathbb{C}$) that satisfies

$$|f(z)| \le A(1 + |z|)^\alpha.$$

for all $z \in \mathbb{C}$.

   (a) Let a, b be complex numbers with $|a| < R, |b| < R, a \neq b$, and let Γ_R be the circle in $\mathbb{C}$ centered at the origin with radius R and oriented counterclockwise. Find

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{f(z)}{(z-a)(z-b)} dz$$

   (b) Show that f is a constant.

6. (a) Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and Ω be a connected, simply connected domain in $\mathbb{C}$. Suppose that $z_0 \in \Omega$ is the only zero of f and $f'(z_0) \neq 0$. Show that

$$z_0 = \frac{1}{2\pi i} \int_\gamma \frac{z f'(z)}{f(z)} dz \qquad\qquad \text{where } i = \sqrt{-1}.$$

   for any piecewise smooth, counterclockwise oriented, simple closed curve $\gamma \subset \mathbb{C}$ containing z_0 in its interior.

   (b) State Rouché's theorem.

   (c) Determine the number of zeros of $P(z) = z^{28} - \sin(z^3) + 10z$ in the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$.
   Hint: Remember that $\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$, where $i = \sqrt{-1}$.