Algebra Qualifying Exam

University of British Columbia

September 1, 2012

1. Consider the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{pmatrix}.$$

   (a) Determine the eigenvalues of A.

   (b) Find eigenvectors correspoding to each of the eigenvalues from part (a).

   (c) Determine

$$\lim_{n\to\infty} A^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2. Let V be a finite-dimensional inner product space, and let $W_1, \ldots, W_k$ be subspaces of V. Let W be the subspace generated (spanned) by $W_1, \ldots, W_k$. Show that $W^\perp = \bigcap_{i=1}^k W_i^\perp$. Here $W^\perp$ denotes the orthogonal complement to W in V with respect to the inner product.

3. For $n \geq 1$ let $V_n = \mathbb{R}[x]^{<n}$ be the vector space of real polynomials of degree less than n. For an integer k define a linear functional $\varphi_k \in V_n^*$ by $\varphi_k(f) = f(k)$.

   (a) Show that $\{\varphi_1, \ldots, \varphi_n\}$ is a basis for the dual space V_n^*.

   (b) Let $D\colon V_3 \to V_3$ be the differentiation map (so that $D(x^2+2x+1) = 2x+2$). Find the matrix of the adjoint $D^*\colon V_3^* \to V_3^*$ in the basis from part (a).

4. Let S_n denote the symmetric group on the n letters $\{1, 2, \ldots, n\}$.

   (a) Let $\tau \in S_5$ be the permutation with cycle representation $(1\,2\,4)$. Find all permutations $\sigma \in S_5$ which commute with τ (recall that two elements σ, τ of a group *commute* if $\tau\sigma = \sigma\tau$).

   (b) Let G be a group, H a subgroup. Recall that the *normalizer* of H in G is the subgroup $N_G(H) = \{x \in G \mid xHx^{-1} = H\}$. Now let H_1, H_2 be subgroups of G that are conjugate to each other. Show that their normalizers $N_G(H_1), N_G(H_2)$ are also conjugate to each other.

(c) Show that all subgroups of S_4 of order 6 are conjugate to one another. (*Hint*: one approach is to apply part (b))

5. For parts (b),(c),(d) of the problem, let f be the polynomial $x^4 + 5x^2 - 3$.
   (a) Show that $x^4 + 2 \in \mathbb{F}_5[x]$ is irreducible.
   (b) Show that $f \in \mathbb{Q}[x]$ is irreducible.
   (c) Let $\Sigma \subset \mathbb{C}$ be the splitting field of f. Determine its degree $[\Sigma : \mathbb{Q}]$.
   (d) Determine the isomorphism class of the Galois group of f.

6. Note: the two parts are independent.
   (a) Let R a commutative ring with identity and let S be a subring of R. Let P be a proper prime ideal of R. Show that $P \cap S$ is a proper prime ideal of S. You need to show that $P \cap S$ is an ideal of S, that it is prime, and that it is not all of S.
   (b) Show that every group of order n can be generated by at most $\log_2 n$ elements.