Pure Mathematics Qualifying Exam
University of British Columbia
September 4, 2009.

Part I

1. (a) Let
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
and verify that for all real numbers x_1, y_1, x_2, y_2 we have
$$(x_1 I + y_1 J)(x_2 I + y_2 J) = (x_1 x_2 - y_1 y_2)I + (x_1 y_2 + x_2 y_1)J.$$
(b) Find A^n for any integer n, if
$$A = \begin{bmatrix} 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}.$$

2. Prove that the equation $XY - YX = I_n$ has no solution (where X, Y are unknown real $n \times n$-matrices, and I_n is the identity matrix).

3. A nonzero matrix A is called nilpotent if there exists a positive integer n such that $A^n = 0$. Two matrices A and B are called similar if they can be obtained from one another by a change of basis.

   (a) Prove that if A is nilpotent and B is similar to A, then B is also nilpotent.

   (b) Find a set of representatives of all equivalence classes of nilpotent 3×3-matrices with complex entries, where we declare two matrices equivalent if they are similar. (You may want to solve this question for 2×2-matrices first).

4. Define the Fourier transform pair to be:
$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \qquad \text{and} \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dx.$$

   (a) Use contour integration to calculate the Fourier transform $\hat{f}(k)$ for
$$f(x) = \frac{1}{(x^2 + a^2)^2},$$
where $a \in \mathbb{R}$ is a constant.

   (b) Calculate the inverse Fourier transform $f(x)$ for
$$\hat{f}(k) = \frac{1}{(k^2 + a^2)^2},$$
where $a \in \mathbb{R}$ is a constant.

5. Use a keyhole-shaped contour to evaluate the integral.

$$I = \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}$$

6. Let D be the circle of radius 4 centred at the point $(0,5)$ in the $x-y$ plane. Find a function $\phi(x,y)$ that satisfies the following restrictions:

   - ϕ is harmonic in the upper half-plane exterior to D;
   - $\phi = 1$ on D;
   - $\phi = 0$ on the x-axis.

   Hint: Consider a conformal map of the form $w = \frac{z+\alpha}{z+\beta}$.

Part II

1. Suppose f is a continuous function on $\mathbb{R}$ such that $|f(x) - f(y)| \geq |x - y|$ for all x and y. Show that the range of f is all of $\mathbb{R}$.

2. For every $a \in \mathbb{R}$, determine whether the integral

$$\iint_D (x^4 + y^2)^a \, dA$$

   is finite, where D is the square $\{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function and assume that f has a local minimum at 0. Prove that there is a disc centered on the y-axis which lies above the graph of f and touches the graph of f at $(0, f(0))$.

4. (a) What is the smallest integer n such that there exists a non-abelian group of order n?

   (b) Give an example of a number $n > 1000$ and not a prime, such that there exists only one group of order n up to isomorphism. How many subgroups does the group in your example have?

5. Let E be the splitting field of the polynomial $(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$.

   (a) Find the degree $[E : \mathbb{Q}]$.

   (b) Find an element $\alpha \in E$ such that $E = \mathbb{Q}(\alpha)$.

   (c) Find the Galois group $\mathrm{Gal}(E/\mathbb{Q})$.

6. Let $I = \{f \in \mathbb{C}[x, y] \mid f(1, 1) = 0\}$. Prove that I is a maximal ideal in the ring $\mathbb{C}[x, y]$.

   Find a minimal set of generators for I.