Fall 2007 Applied Qualifying Exam

Part I

1. Suppose f is a differentiable real-valued function such that $f'(x) > f(x)$ for all x and $f(0) = 0$. Prove that $f(x) > 0$ for all $x > 0$.

2. Let $x_0 = 0$ and

$$x_{n+1} = \frac{1}{2 + x_n}$$

for $n = 0, 1, 2, \ldots$. Prove that $x_\infty = \lim_{n\to\infty} x_n$ exists and find its value.

3. Show that there is no nonzero polynomial $P(u, v)$ in two variables with real coefficients such that

$$P(x, \cos x) = 0$$

holds for all real x.

4. Let $\mathbf{A}$ be the $n \times n$ matrix with all diagonal entries s and all off-diagonal entries t. For which complex values of s and t is this matrix not invertible? For each of these values, describe the nullspace of $\mathbf{A}$ (including its dimension).

5. Show that the matrix

$$\begin{bmatrix} 0 & 5 & 1 & 0 \\ 5 & 0 & 5 & 0 \\ 1 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

has two positive and two negative eigenvalues.

6. Given an inner product $(\mathbf{x}, \mathbf{y})$ for vectors $\mathbf{x}$ and $\mathbf{y}$ over the field of real numbers, prove the Cauchy-Schwartz inequality

$$|(\mathbf{x}, \mathbf{y})| \leq |\mathbf{x}||\mathbf{y}|$$

for all $\mathbf{x}$ and $\mathbf{y}$. *Hint:* consider

$$|\mathbf{x} + t\mathbf{y}|^2$$

for real t. Using the above inequality, prove the triangle inequality

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|.$$

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Part II

1. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin 3x dx}{x^2 + 2x + 3}.$$

2. Let f be analytic in the disk $|z| < R$ where $|f(z)| < M$. Find a bound for

$$|f'(re^{i\theta})|$$

in terms of R, M and $r < R$.

3. Consider the function

$$f(z) = (z^4 - 9)^{1/4}.$$

Define a branch that is analytic for all $|z| < \sqrt{3}$ and that has the value

$$f(0) = 9^{1/4} e^{i\pi/4}.$$

Sketch your branch cuts. For the branch you have chosen, calculate $f(i)$.

4. Consider the differential equation

$$2xy'' + y' + x^2 y = 0$$

Find two linearly independent Frobenius series solutions to this equation, *i.e.* solutions in the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

where r and the a_n are to be determined. It is sufficient to find the first three nonzero terms in each series. What kind of initial conditions at $x = 0$ are appropriate to determine a solution for $x > 0$?

5. The function $T(x, y)$ satisfies the equation

$$T_{xx} + T_{yy} = 0$$

in the semi-infinite strip $x \geq 0$ and $0 \leq y \leq 1$ with the boundary conditions:

$$\begin{aligned} T(x,y) &\to 0 \text{ as } x \to \infty \\ T_y &= 0 \text{ on } y = 0 \text{ and } y = 1 \text{ for all } x \\ T(0,y) &= y \text{ for } 0 < y < 1 \end{aligned}$$

Find a series solution to this problem. *Hint:* begin by seeking separable solutions. Discuss briefly the behaviour of the solution near the origin. Is the solution continuous there? Differentiable?

6. A population model evolves according to the map

$$x_{n+1} = f(x_n) = x_n e^{r(1-x_n)}$$

with real parameter $r > 0$. Find the fixed points and their stability. Show that a bifurcation occurs at $r = 2$. What kind of bifurcation is it? Sketch the graph of the map for $r > 1$ and describe the dynamics qualitatively.