## Applied Math Qualifying Exam: Sept. 11, 2004

### Part I

1. Let $0 < b < a$. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos(\theta))^2}.$$

2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $a, b, c, d > 0$. Show that $A$ has an eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, with $x, y > 0$.

3. Consider the following initial-boundary-value problem for $u(x,t)$:

$$\begin{cases} u_t + u_{xxxx} = 0, \quad 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = u_{xxx}(0,t) = u_x(\pi,t) = u_{xxx}(\pi,t) = 0, \quad t > 0 \\ u(x,0) = \cos^2(x), \quad 0 < x < \pi \end{cases}.$$

   (a) The solution tends to a steady-state, $v(x) = \lim_{t\to\infty} u(x,t)$. Find $v(x)$.

   (b) Find the solution $u(x,t)$.

   (c) How much time does it take for $u(x,t)$ to get within $10^{-2}$ of the steady-state for all $x \in (0,\pi)$?

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

   (a) For which $x$ does the series converge absolutely?

   (b) On which intervals does it converge uniformly?

   (c) Is $f$ continuous wherever the series converges?

   (d) Is $f$ bounded?

5. Suppose $f(z)$ is analytic on a connected region $\Omega \subset \mathbb{C}$. Show that $|f(z)|^2$ is harmonic on $\Omega$ if and only if $f$ is constant.

6. Let $V$ be the vector space of continuous, real-valued functions on the interval $[0,\pi]$, with the inner-product

$$\langle f, g \rangle := \int_0^{\pi} f(x)g(x)dx.$$

   (a) Find an orthonormal basis for the subspace $S :=$ span $\{1, \sin(x)\}$.

   (b) Compute the distance of $\sin^2(x)$ from $S$.

## Part II

1. Consider the vector field $\mathbf{F}(x, y, z) = (yz + x^4)\hat{\mathbf{i}} + (x(1+z) + e^y)\hat{\mathbf{j}} + (xy + \sin(z))\hat{\mathbf{k}}$. Let $C$ be a circle of radius $R$ lying in the plane $2x + y + 3z = 6$. What are the possible values of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$?

2. A forced mass-spring system is governed by the following ODE for $y(t)$:

$$y'' + ky = f(t) \qquad (*)$$

where $k > 0$ is a constant, and $f$ is a smooth, odd, $T$-periodic function.

(a) Find the general solution when $f = 0$.

(b) By expanding $f$ in a Fourier series, find a formal (i.e. infinite series) particular solution of (*).

(c) Under what conditions on $f$ and $k$ will the system exhibit resonance?

3. Show that $2e^{-z} - z + 3$ has exactly one root in the right half-plane $\{z \in \mathbb{C} \mid Re(z) > 0\}$.

4. Let $a, b, c, d$ be real numbers, not all zero. Find the eigenvalues of the following $4 \times 4$ matrix and describe the eigenspace decomposition of $\mathbb{R}^4$:

$$\begin{pmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{pmatrix}$$

.

5. Define $f(x) = x^2$ for $-\pi < x \leq \pi$, and extend it to be $2\pi$-periodic.

(a) Find the Fourier series of $f$.

(b) Use (a) to evaluate $\sum_{j=1}^{\infty} \frac{(-1)^j}{j^2}$ (state clearly any theorems you use).

6. Consider the following PDE for $u(x, t)$:

$$u_t - au_{xx} - bu + cu^3 = 0, \quad -\infty < x < \infty, \quad t > 0$$

($a, b, c > 0$ are constants).

(a) Use scaling to reduce the problem to the form

$$w_t - w_{xx} - w + w^3 = 0, \quad -\infty < x < \infty, \quad t > 0, \qquad (*)$$

(b) Suppose $w(x, t)$ is a smooth solution of $(*)$ with $w_x(x, t), w_t(x, t) \to 0$ as $x \to \pm\infty$. Show that the quantity

$$\int_{-\infty}^{\infty} \left\{ \frac{1}{2} w_x^2(x, t) + \frac{1}{4}(w^2(x, t) - 1)^2 \right\} dx$$

(if it is finite) is a non-increasing function of time.

(c) Suppose further that

$$w(x,t) \to \begin{cases} -1 & x \to -\infty \\ 1 & x \to +\infty \end{cases} .$$

Suppose the solution tends to a steady-state, $v(x) = \lim_{t\to\infty} w(x,t)$. Find the form of $v(x)$.