Qualifying Exam Problems: Algebra

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1. (10 points) (a) Show that any subgroup of a group of order 341 is abelian.
   (b) Let A be an infinite set and let S_A be the permutation group on A. Consider the following subsets of S_A.
   $$H = \{\sigma \in S_A \mid \sigma \text{ moves at most five elements of } A\}$$
   $$K = \{\sigma \in S_A \mid \sigma \text{ moves finitely many elements of } A.\}$$
   Which of these subsets is a group? Justify your answer.
   (c) Let p be a prime and S_p be the symmetric group on p elements. Show that G has $(p-2)!$ p-Sylow subgroups and deduce the congruence $(p-1)! \equiv -1 \mod p$.

2. (10 points) (a) Let K be a field and let $f(x)$ in $K[X]$ be an irreducible polynomial of degree 7 with splitting field M. Suppose that the Galois group $\mathrm{Gal}(M/K) \simeq S_7$, Let α be a root of f and put $L = K(\alpha)$. Prove that if E is an extension of K such that $K \subseteq E \subseteq L$, then either $E = K$ or $E = L$.
   (b) Let L be the splitting field of $p(X) = (X^3 - 2)(X^2 - 3)$ over $\mathbb{Q}$ and let G be the Galois group of $p(X)$ over $\mathbb{Q}$. Find the degree $[L : \mathbb{Q}]$.
   (c) Express the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ as a direct product of two nontrivial groups.

3. (10 points) (a) Let R be a unique factorization domain and let K be the quotient field of R. An element $z \in K$ is said to be integral over R if there exists a monic polynomial $f \in R[x]$ such that $f(z) = 0$. Prove that if z is integral over R, then z is in R.
   (b) Let t_1, t_2, t_3 be the roots of the polynomial $X^3 + 3X - 1$ over $\mathbb{Q}$. Find the minimal polynomial of $\frac{1}{t_3}$.
   (c) Let $a, b \in \mathbb{Z}$. Show that if 5 divides $a^2 - 2b^2$, then 5 divides both a and b.
   (d) Let $R = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ and let $M = \{a + b\sqrt{2} \in R : 5 \mid a \text{ and } 5 \mid b\}$. Show that M is a maximal ideal in R and compute the order of the field R/M.

4. (10 points) Let $A \in M_{n,n}(\mathbb{R})$ be a matrix of rank $n-1$. Let $L_A : M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$ be the function given by $L_A(B) = A \cdot B$.
   (a) Show that L_A is a linear map.
   (b) Find the dimension of the image of L_A.
   (c) Find a basis for the image of L_A.

5. (10 points) Let $k \in \mathbb{N}$, let $A_1, \dots, A_k \in M_{n,n}(\mathbb{R})$ and let
   $$B = \sum_{i=1}^{k} A_i \cdot A_i^t,$$
   where for each matrix C, we denote by C^t its transpose.
   (a) Prove that B is a symmetric matrix.
   (b) Prove that B is a positive definite matrix, i.e. for each vector $v \in M_{n,1}(\mathbb{R})$, the dot product $\langle Bv, v\rangle$ is nonnegative.
   (c) Prove that $\det(B) \geq 0$.

6. (10 points) Solve the following system of linear equations:
   $$\left\{\begin{array}{ll} x_1 + x_2 + x_3 + x_4 & = 0 \\ 2x_1 + 4x_2 + 8x_3 + 10x_4 & = 2 \\ -2x_1 - x_2 + x_3 + 2x_4 & = 1 \\ -10x_1 - 8x_2 - 4x_3 - 2x_4 & = 2 \end{array}\right. .$$
   Explain your answer.