Differential Equations Qualifying Exam

University of British Columbia

January 5, 2013

1. Let S_1 and S_2 be subspaces of a vector space.
   (a) Give an example to show that in general $S_1 \cup S_2$ need not be subspace.
   (b) If $S_1 \cup S_2$ is a subspace, prove that $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

2. Let V be an n-dimensional linear subspace of $\mathbb{R}^N$ $(1 \le n \le N)$. We denote by $\langle v, w \rangle$ the ordinary dot product of vectors $v, w \in \mathbb{R}^N$.
   (a) If $\{v_1, \ldots, v_n\}$ is a basis for V, define for each i the linear map $\nu^i(v_j) = \delta^i_j$ where $\delta^i_j = 0$ if $i \neq j$ and $\delta^i_i = 1$. This definition on the basis vectors then uniquely determines ν^i as an element of V^*, the dual space of V. Prove that $\{\nu^1, \ldots, \nu^n\}$ is a basis for V^*.
   (b) Given $v \in V$, let $v^* : V \to \mathbb{R}$ be the linear map $v^*(w) = \langle w, v \rangle$. Show that the map $v \to v^*$ defines a linear isomorphism from V to V^*.
   (c) We extend the dot product to V^* by requiring $\langle v^*, w^* \rangle = \langle v, w \rangle$ for $v, w \in V$. If $\{v_1, \ldots, v_n\}$ is a basis for V, let $g_{ij} = \langle v_i, v_j \rangle$ for $1 \le i, j \le n$. Show that $\langle \nu^i, \nu^j \rangle = g^{ij}$ where the matrix (g^{ij}) is the inverse matrix of (g_{ij}).
   (d) Determine the change of basis matrix from $\{v_1^*, \ldots, v_n^*\}$ to $\{\nu^1, \ldots, \nu^n\}$.

3. (a) Let A be an $n \times n$ real matrix. Show that if $A^k = 0$ for some positive integer k and A is not the zero matrix, then A cannot be diagonalized. Show that 0 is an eigenvalue of A.
   (b) Let V be a vector space, and suppose that $v_1, \ldots, v_n$ are eigenvectors of a linear map $T : V \to V$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $v_1, \ldots, v_n$ are linearly independent.

4. Find the general solution to the system of equations
$$\boldsymbol{x}' = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} \boldsymbol{x}.$$

5. Let $y(x)$ be a smooth curve over the interval $(-\pi, \pi)$. On the interval $(-\pi, 0)$, the curve passes through P: $\left(-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right)$ and the normal to $y(x)$ at any point on the curve passes through the origin. On the interval $[0, \pi)$, the function $y(x)$ satisfies the second-order ODE $y'' = -y - x$. Find $y(x)$.

6. The steady-state temperature distribution inside a circular metal disc, whose edge has a prescribed temperature profile, is determined by Laplace's equation:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0,$$

with boundary condition $T(a, \theta) = p(\theta)$. Here r and θ are the polar coordinates, a is the radius of the disc, and $p(\theta)$ is the prescribed edge temperature around the disc. Using separation of variables, one can derive the following solution to Laplace's equation on the disc:

$$T(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta, \quad \text{for } 0 \leq r \leq a, -\pi < \theta \leq \pi.$$

(a) Determine the coefficients A_n and B_n so that the solution satisfies the boundary condition at the edge $T(a, \theta) = p(\theta)$.

(b) Use the above solution to show that temperature at the center ($r = 0$) is equal to the average around the edge: $T_0 = T(0, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta)\, d\theta$.

(c) Use this *mean value property* to show that the maximum and minimum temperatures on the disc occur on the boundary.

(d) The above property holds for elliptic partial differential equations in general, and is known as the *maximum principle*. Use this principle to show that the solution to Poisson's equation,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = q(r, \theta),$$

with the boundary condition $T(a, \theta) = p(\theta)$, is unique.