# Applied Mathematics Qualifying Exam
## January 7, 2006

## Part I

PROBLEM 1. Find the critical points of

$$f(x,y) = x^2 + 2xy + 2y^2 - \frac{1}{2}y^4$$

and classify each one as a local minimum, local maximum, or saddle point.

PROBLEM 2. Consider radial symmetric diffusion of a chemical species with concentration $c(r,t)$ in a circular cylindrical domain with an insulating boundary. An appropriate model for $c(r,t)$ is

$$c_t = D\left(c_{rr} + \frac{1}{r}c_r\right), \qquad 0 \le r \le a, \quad t > 0,$$
$$c_r = 0 \quad \text{on } r = a,$$
$$c \text{ is bounded as } r \to 0; \qquad c(r,0) = f(r), \quad 0 \le r \le a.$$

Here $D$ and $a$ are positive constants.

(a) Calculate the steady-state solution for this problem corresponding to the limiting behavior of $c(r,t)$ as $t \to \infty$.

(b) Determine an eigenfunction series representation for the time-dependent solution.

(c) What would the corresponding steady-state solution be in the annulus $b < r < a$, with $0 < b < a$, if there was no flux of $c$ across $r = a$ and $r = b$?

PROBLEM 3. Evaluate the following integral using the method of residues, carefully justifying each step:

$$\int_0^\infty \frac{x^2\,dx}{x^4 + 5x^2 + 6}.$$

PROBLEM 4. Let $V$ be the vector space of polynomials in one variable of degree at most $n$, with real coefficients. Given distinct real numbers $a_0, a_1, \ldots, a_n$, show that any polynomial $f(x) \in V$ can be expressed in the form

$$f(x) = c_0(x + a_0)^n + c_1(x + a_1)^n + \ldots c_n(x + a_n)^n$$

for some $c_i \in \mathbb{R}$.

PROBLEM 5. Consider the complex multi-valued function

$$f(z) = (z^3 + z^2 - 6z)^{1/2}.$$

(a) Find a set of branch cuts of the complex plane such that on the complement of these cuts $f(z)$ can be defined as a single-valued function. Moreover, the cuts should be such that if we require $f(-1) = -\sqrt{6}$, then such a single-valued $f(z)$ is unique.

(b) With the branch cuts as above, choose an arbitrary point $p$ lying on one of the cuts, but $p$ should not be a branch point, and describe the limiting behavior of $f(z)$ as $z$ approaches $p$ along different paths.

PROBLEM 6. Let $f(x)$ be a continuous function for all $x$ and let $\epsilon > 0$. Does the following limit exist? (if yes, then prove that the limit exists; if no then provide a counterexample).

$$\lim_{\epsilon \to 0+} \left( \int_{-1}^{-\epsilon} \frac{f(x)}{x}\, dx + \int_{\epsilon}^{1} \frac{f(x)}{x}\, dx \right).$$

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## Part II

PROBLEM 1. Let $S$ be the hemisphere $\{x^2 + y^2 + z^2 = 1, z \geq 0\}$ oriented with $\mathbf{N}$ pointing away from the origin. Use the divergence theorem to evaluate the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where

$$\mathbf{F} = (x + \cos(z^2))\mathbf{i} + (y + \ln(x^2 + z^5))\mathbf{j} + \sqrt{x^2 + y^2}\mathbf{k}.$$

PROBLEM 2. Consider the following predator-prey model for $x = x(t) \geq 0$ and $y = y(t) \geq 0$:

$$\frac{dx}{dt} = -x + xy\,, \qquad \frac{dy}{dt} = -xy + ry\left(1 - \frac{y}{k}\right)\,.$$

Here $r > 0$ and $k > 1$ are constants.

(a) Show that the system has critical points at $(0, 0)$, $(0, k)$, and at some $(x_*, y_*)$, with $x_* > 0$ and $y_* > 0$. Calculate $x_*$, $y_*$ explicitly.

(b) Classify the type and stability of each critical point (saddle, stable node, etc.).

(c) Show that $(x_*, y_*)$ must be a stable spiral point when $k$ is very large, and give a sketch of the phase plane in this case.

PROBLEM 3. The following system of three nonlinear algebraic equations is to be solved for $x, y, z$ in terms of $u, v, w$:

$$u = x + y^2 + z^3\,; \qquad v = x^3 + y + z^2\,; \qquad w = x^2 + y^3 + z\,.$$

Prove or find a counterexample to the statement that there is a unique solution near $(x, y, z) = (0, 0, 0)$ if $u, v, w$ are all small.

PROBLEM 4. All matrices in this problem have real coefficients and size $n \times n$.

It is well-known that a positive definite symmetric matrix $A$ has a square root $Q$ in the sense that

$$A = QQ^T.$$

Use this fact to show that if $A$ and $B$ are positive definite symmetric matrices, then the eigenvalues of $AB$ are real and positive.

(continued on back)

PROBLEM 5. Consider the lens-shaped region $\mathcal{L}$ formed from the intersection of the disks $|z| < 1$ and $|z-1| < 1$.

(a) Find a bounded harmonic function $\phi$ inside $\mathcal{L}$ that satisfies $\phi = 0$ on $|z-1| = 1$ and $\phi = 1$ on $|z| = 1$. (Hint: First find a Möbius transformation of the form

$$z \mapsto \frac{az+b}{cz+d}$$

that maps $\mathcal{L}$ onto a portion of the upper half-plane).

(b) Explain why there is no Möbius transformation that maps the lens-shaped region $\mathcal{L}$ onto the unit disk.

PROBLEM 6. Consider the function $f(x) = \cos(\nu x)$, defined on $0 \le x \le \pi$, where $\nu$ is an arbitrary real number.

(a) Calculate the Fourier cosine series of $f(x)$. In what sense does the series converge? (Hint: In calculating the Fourier cosine coefficients it is helpful to use the complex representation of $\cos(x)$).

(b) From your explicit Fourier cosine series, and for $\nu$ not an integer, derive that

$$\pi \cot(\pi\nu) = \frac{1}{\nu} + \sum_{n=1}^{\infty} \left( \frac{1}{\nu+n} + \frac{1}{\nu-n} \right) .$$

(c) By integrating the expression above derive the identity

$$\frac{\sin(\pi\theta)}{\pi\theta} = \prod_{n=1}^{\infty} \left( 1 - \frac{\theta^2}{n^2} \right) , \qquad 0 < \theta < 1 .$$