Applied Qualifying Exam
January 10, 2004.
Part I

1. For what values of the real constants a and b is

$$f(z) = axy + i(x^2 + by^2)$$

analytic? Here we have used $z = x + iy$.

2. Find the distance from the ellipse

$$\frac{x^2}{4} + y^2 = 1$$

to the straight line $x + y = 4$.

3. Let A be an $n \times n$ matrix with complex entries. An $n \times n$-matrix B is called a *square root* of A if $B^2 = A$. Suppose A is non-singular and has n distinct eigenvalues. How many square roots does A have?

4. Let f be a real function on $[0, 1]$ having the following property: for any real y, the equation $f(x) - y = 0$ has either no roots, or exactly two roots. Prove that f cannot be continuous at every point in the interval $[0, 1]$.

5. Define a sequence $x_1, x_2, \ldots$ recursively by $x_0 = c$, $x_1 = 1 - c$, and

$$x_{n+2} = 2.5x_{n+1} - 1.5x_n$$

for $n \geq 1$. For what values of c does the sequence $\{x_n\}$ converge? If it converges, what is the value of $\lim_{n\to\infty} x_n$?

6. Consider the system in the plane

$$\frac{dx}{dt} = y - x^3, \qquad \frac{dy}{dt} = x - y^2.$$

   (a) Find all fixed points of this system. Use linearized stability analysis to determine which fixed points are stable.

   (b) Sketch the phase portrait (solution curves in the $x - y$ plane).

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Part II

1. Consider the following partial differential equation for $u(x, t)$:

$$u_t + \alpha u_{xxxx} + \beta u_{xx} + \gamma u u_x = 0 \tag{1}$$

where $u(x, t)$ is $L-$periodic in x for all t. The parameters α, β and γ are positive.

   (a) Use scaling to minimize the number of essential parameters.

   (b) Show that for smooth solutions $u(x, t)$ of (1)

$$M = \int_0^L u(x, t)dx$$

   is constant in time.

2. Introduce new coordinates into the plane quadrant $x > 0$, $y > 0$ through the transformation:

$$\xi = x^2 y; \ \eta = xy^2.$$

   (a) Determine x and y as functions of ξ and η.

   (b) Compute the Jacobian matrices

$$\mathbf{A} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$$

   (c) Compute and simplify **AB**. Comment on the result.

3. Are the following statements true? In each case give a proof or a counterexample. Assume that A and B are $n \times n$-matrices with real entries and $n \geq 2$.

   (a) If $\det(A) = \det(B) = 1$ then $A + B$ is non-singular.

   (b) If A and B are symmetric matrices all of whose eigenvalues are strictly positive, then $A + B$ is non-singular.

4. Evaluate the integral

$$\int_0^\infty \frac{\cos x}{x^2+9}dx.$$

5. A function is said to be *even* if $f(x) = f(-x)$ for all x. Let $\mathcal{V}$ be the vector space of all even polynomials $p(x)$ of degree less than or equal to $2n$. Let $\mathbf{A}$ be the operator

$$\mathbf{A} = \frac{d^2}{dx^2}$$

acting on $\mathcal{V}$.

   (a) Prove that 0 is the only eigenvalue of $\mathbf{A}$. What is the corresponding eigenspace?

   (b) Prove that the operator mapping the polynomial $p(x)$ into the polynomial

   $$q(x) = p(x+1) + p(x-1)$$

   defines a linear mapping $\mathbf{B}$ of $\mathcal{V}$ into itself.

   (c) Does $\mathbf{B}$ commute with $\mathbf{A}$?

6. Consider the following PDE problem for $u(x,t)$ on the domain $x \geq 0$ and $t \geq 0$:

$$\begin{aligned} u_t &= u_{xx} \quad \text{for } x > 0,\ \ t > 0 \\ u(x,0) &= 0 \quad \text{for } x \geq 0 \\ u(0,t) &= \sin \omega t \quad \text{for } t \geq 0 \\ \lim_{x\to\infty} u(x,t) &= 0 \quad \text{for all } t \geq 0 \end{aligned}$$

where ω is a given positive constant, the angular frequency of the forcing at the boundary. As $t \to \infty$ the solution u tends to a limiting solution that has angular frequency ω. Determine an explicit formula for this limiting solution.