The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
January 2021

In the real and complex analysis parts of this exam, please state carefully any results that you use in your arguments

1. (3 points) True or false: There exists an infinite set X for which the normed space

$$\mathcal{B}(X) = \big\{f : X \to \mathbb{R} : f \text{ is bounded}\big\}$$

is separable, under the sup norm.

2. (3 points) True or false: Any Lipschitz function $f : \mathbb{Q} \to \mathbb{R}$ extends uniquely to a continuous function $g : \mathbb{R} \to \mathbb{R}$.

3. (5 points) True or false: There exists a sequence $\{f_n : n \geq 1\} \subseteq \mathcal{C}[0,1]$ with $||f_n||_\infty \leq 1$ for which no subsequence of

$$F_n(x) = \int_0^x f_n(t)\, dt$$

is uniformly convergent on $[0,1]$. Here $\mathcal{C}[0,1]$ denotes the space of real-valued continuous functions on $[0,1]$.

4. (5 points) True or false: Let $V = \bigcup_{n=1}^\infty W_n$ be an infinite-dimensional normed vector space, where each W_n is a finite-dimensional subspace of V. Then V cannot be complete.

5. (5 points) True or false: Every subset of a metric space can be written as the intersection of open sets.

6. (5 points) True or false: Every bounded continuous function on $\mathbb{R}$ is uniformly continuous.

7. (4 points) True or false: The limit of a point-wise convergent sequence of Riemann integrable functions on $[0,1]$ must be Riemann integrable.

Complex analysis

8. (a) (3 points) Compute

$$\oint_\Gamma \left(\frac{1}{(z-\mathrm{i})^2} + 2\bar{z} \right) \mathrm{d}z$$

where Γ is the positively positively oriented triangle with corners $0, 1, 1+\mathrm{i}$.

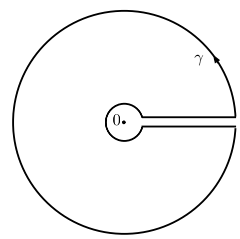
(b) (2 points) Let $\mathcal{C}$ be the positively oriented unit circle and let g be a continuous, complex-valued function defined in a neighbourhood of $\mathcal{C}$. Prove that

$$\overline{\oint_{\mathcal{C}} g(z) dz} = \oint_{-\mathcal{C}} \overline{g(z)} \frac{dz}{z^2},$$

where $-\mathcal{C}$ denotes the negatively oriented circle.

(c) (5 points) Let $0 < \alpha < 1$. Use the proposed curve γ to compute

$$\int_0^\infty \frac{x^\alpha}{1+x^2} \mathrm{d}x.$$



9. (a) (3 points) Consider the mapping $\theta : \mathbb{C} \to \mathbb{C}$ given by $\theta(x+\mathrm{i}y) = y + \mathrm{i}x$. Let $\Omega \subset \mathbb{C}$ be open and such that $\theta(\Omega) = \Omega$. Let f be a holomorphic function in Ω. Let $g : \Omega \to \mathbb{C}$ be given by $g(z) = \overline{f(\theta(z))}$. Prove that g is holomorphic in Ω.

(b) (3 points) Consider the Taylor series of

$$\frac{6}{b - 2z^2 + z^3} = 3 + \sum_{n=1}^{\infty} a_n z^n$$

around $z_0 = 0$. Compute b, a_1, a_2 and find a recursion relation of the form

$$a_n = \alpha a_{n-1} + \beta a_{n-2} + \gamma a_{n-3}$$

that is valid for all $n \geq 3$.

(c) (4 points) Recall that $n!! = n(n-2)(n-4)\cdots$, for example $8!! = 8 \cdot 6 \cdot 4 \cdot 2$. Show that the unique analytic solution of the differential equation

$$f''(z) - zf'(z) - f = 0$$

such that $f(0) = \mathrm{i}$ and $f'(0) = 0$ is given by $f(z) = \mathrm{i} + \sum_{j=1}^{\infty} \frac{\mathrm{i}}{(2j+1)!!} z^{2j+1}$.

10. (a) (4 points) Let $r = \frac{1}{\sqrt{3}}$, $\Omega = \{|z| < \frac{2r}{3} = \frac{2}{3\sqrt{3}}\}$ and γ be the circle centred at the origin of radius r, oriented positively.
(i) Prove that for any $w \in \Omega$, there is a unique solution ζ in the disc $\{|z| < r\}$ of $z^3 + z = w$.
(ii) Denote the solution of part (i) by $\zeta = f(w)$. Prove that

$$f(w) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{z(3z^2+1)}{z^3 + z - w} dz.$$

(b) (3 points) Let f be entire and for which there is $\alpha > 0$ such that $|f(z)| \leq B|z|^\alpha$ for $|z|$ sufficiently large. Prove that f is a polynomial.

(c) (3 points) Determine all functions f holomorphic in the unit disc such that

$$f(1/n) = 1/n^2$$

for all n large enough. *Hint:* Consider the function $g(z) = f(z) - z^2$.