The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
September 3, 2019

Real analysis

1. (10 points) Are the following true or false? If true, give a proof. If false, provide a counterexample and show that your counterexample is correct.
   (a) Let $v\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Then $v \cdot \nabla f$ is continuous.
   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $\frac{d^2 f}{dxdy}(0,0)$ and $\frac{d^2 f}{dydx}(0,0)$ both exist, then they must be equal.

2. (10 points) Let $f : [0,1] \to \mathbb{R}$ be bounded. Suppose that f is continuous at every point $x \in [0,1]\backslash\{\frac{1}{n} : n \in \mathbb{N}\}$. Prove that f is Riemann integrable on $[0,1]$.

   Note: while it is true that a function g is Riemann integrable provided it is continuous except on a set of measure 0, you cannot use this fact unless you prove it (doing this is not recommended).

3. (10 points) Let (M,d) be a compact metric space and let $C(M)$ be the set of continuous functions $f\colon M \to \mathbb{R}$. Define a metric ρ on $C(M)$ by $\rho(f,g) = \sup_{x\in M} |f(x) - g(x)|$. For each $x \in M$, define the function $f_x\colon M \to \mathbb{R}$ by
$$f_x(y) = d(x,y).$$
   (a) Prove that $f_x \in C(M)$ for each $x \in M$.
   (b) Let $A \subset C(M)$ be the algebra generated by the constant functions and the functions $\{f_x\colon x \in M\}$. Prove that A is dense in $C(M)$.
   (c) Prove that $C(M)$ is separable.

Complex analysis

4. (a) (2 points) Express the principal value of $\left(-1-i\sqrt{3}\right)^{i3\pi}$ in the form of $a+ib$.

   (b) (2 points) Evaluate the contour integral $\displaystyle\int_C e^{\pi z}dz$ where C is a semi-circular arc starting at $z_0 = i$ and ending at $z_1 = i/2$, oriented counterclockwise.

   (c) (3 points) Determine, with complete justification, the domain of analyticity of the function

   $$f(z) = \mathrm{Log}(z^2 + 2z + 3),$$

   where Log denotes the principal branch of the logarithm. Sketch this domain and find the derivative of f at $z = -1$.

   (d) (3 points) Calculate $\oint_C e^{1/z}\frac{1}{1-z}dz$ where C is the circle $\{z : |z| = 1/2\}$ oriented counterclockwise.

5. (a) (5 points) Suppose $f(z)$ and $\overline{f(z)}$ are both analytic in a domain D. Prove that f must be constant in D.

   (b) (5 points) Show: If f is entire and $|f(z)| \to \infty$ as $|z| \to \infty$, then f must have at least one zero.

6. (a) (2 points) Find and classify the singularities of $f(z) = \dfrac{\cot(\pi z)}{z^2}$.

   (b) (3 points) Calculate the residue of $f(z) = \dfrac{\cot(\pi z)}{z^2}$ at each of its singularities.

   (c) (3 points) The sum $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ can be evaluated by integrating $f(z) = \dfrac{\cot(\pi z)}{z^2}$ over a suitable contour Γ_N and taking $N \to \infty$. Draw Γ_N and mark the singularities on your diagram. What does Cauchy residue theorem say when applied to Γ_N?

   (d) (2 points) State what estimates are required to perform the evaluation of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ (you need not prove the estimates). What is the value of the infinite sum?