The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
September 2019

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1. (15 points) (a) Work over the complex numbers. Let $A = \begin{pmatrix} 3 & -2 \\ 8 & -5 \end{pmatrix}$. Find the eigenvalues and state their geometric and algebraic multiplicities.

   (b) Is the matrix above diagonalizable? Explain your answer.
   In either case, write down a similarity transform putting A in Jordan normal form.

   (c) Let B be a real matrix with characteristic polynomial $(x+2)^2(x-3)^2$. What are the possible Jordan normal forms of B? To avoid repetition, give your answers with eigenvalues sorted from smallest-in-magnitude to largest-in-magnitude, and if two Jordan forms J_1 and J_2 happen to be similar matrices, give only one. You may omit 0-entries if you wish.

2. (15 points) (a) Let P_n be the $\mathbb{R}$-vector space of polynomials of degree at most n with real coefficients. Let $S = \{p_1, \ldots, p_{n+1}\} \subseteq P_n$ be a set of $n+1$ polynomials, satisfying $p_i(0) = 0$ for all i. Either prove S is linearly dependent, or give an example to show S may be linearly independent.

   (b) Let $\vec{u}$ and $\vec{v}$ be elements of a real inner-product space. Suppose that
   $$|\vec{u} + \vec{v}| = |\vec{u}| + |\vec{v}|.$$
   Show that $\vec{u}$ and $\vec{v}$ are linearly dependent. Name, or otherwise state clearly, any theorems that you use.

3. (15 points) Let k be a field. Consider the 2×2 matrix
   $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
   with entries in k.

   (a) Let $z \in k$ satisfy the *sector equation*: $bz^2 + (a-d)z - c = 0$. Show that $\vec{v} = \begin{pmatrix} 1 \\ z \end{pmatrix}$ is an eigenvector of A and determine the corresponding eigenvalue.

   (b) Now consider the quotient ring $R = k[\epsilon]/(\epsilon^2)$. All elements of R may be written uniquely in the form $x + \epsilon y$ where $x, y \in k$. Determine all solutions $z \in R$ to the equation
   $$\epsilon z^2 - z - 1 = 0.$$

   (c) Let the ring R be as in the previous part. Find a vector $\begin{pmatrix} r \\ s \end{pmatrix}$ of elements in R such that the ideal generated by $\{r, s\}$ is all of R and such that
   $$\begin{pmatrix} 1 & \epsilon \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \lambda \begin{pmatrix} r \\ s \end{pmatrix}$$
   for some $\lambda \in R$. Hint: try solving an equation like the sector equation from part (a).

4. (15 points) Let R be a commutative ring and let $f : R \to R$ be a surjective ring homomorphism.

   (a) Let f^n denote the composite of f with itself n times. Suppose there exists some integer $m \geq 1$ such that $\ker(f^{m+1}) \subset \ker(f^m)$. Prove that f is injective.

   (b) Give an example of a ring R and a homomorphism $f : R \to R$ that is surjective but not injective (you do not have to provide proof).

5. (15 points) Let p be a prime number. Let k be a field of characteristic p and $\bar{k}$ be an algebraic closure of k. Let $c \in k$ be an element. Consider the polynomial

$$f(x) = x^p - x + c.$$

   a. Suppose $\alpha \in \bar{k}$ is a root $f(\alpha) = 0$. Determine $f(\alpha + 1)$.

   b. Prove that if f does not split over k, then f is irreducible over k.

6. (15 points) Let $D = D_7$ denote the dihedral group of order 14. This group has a presentation $D = \{a, b \mid a^2 = b^7 = abab = e\}$.

   (a) Let $i \in \{0, \ldots, 6\}$. Determine the order of the element ab^i in D, in terms of i.

   (b) Write down all elements of order 7 in D.

   (c) Consider the set G of all group automorphisms $\phi : D \to D$. The set G forms a group under composition. What is the order of G?

   (d) Describe a homomorphism $\phi : D \to D$ such that $\phi \neq \mathrm{id}_D$ but such that $\phi \circ \phi \circ \phi = \mathrm{id}_D$.