The University of British Columbia
Department of Mathematics
Qualifying Examination — Analysis
September 4, 2018

Give careful statements of theorems you are using.

I. Real Analysis

Do 3 of the following 4 questions. Indicate clearly which 3 are to be graded.

1. (10 points) Let $\vec{C}$ be a smooth simple closed curve with positive orientation enclosing a region D in the plane. Suppose D has area 5 and centroid $(3, 2)$.
(a) Find
$$\int_{\vec{C}} (3y + x^2)dx + 2xydy.$$
(b) If $T(u, v) = (u - v, u + 2v)$, find the area of $D' = T(D) = \{T(u, v) : (u, v) \in D\}$.
Hint: Recall that the centroid of a region D with area A is the point
$$(\bar{x}, \bar{y}) = \frac{1}{A}\Big(\iint_D x\ dxdy, \iint_D y\ dxdy\Big).$$

2. (10 points) Let $\mathcal{P} = \Big\{\sum_{n=1}^{N} a_n x^n : a_n \in \mathbb{R}, N \in \mathbb{N}\Big\}$, be a set of polynomial functions on $[0, 1]$. (Note: there is no constant term!)
(a) State the Weierstrass approximation theorem.
(b) Prove that if $f : [0, 1] \to \mathbb{R}$ is continuous and $f(0) = 0$, then f is a uniform limit of a sequence of polynomials in $\mathcal{P}$. Hint: You may use (a).
(c) Assume $g : [0, 1] \to \mathbb{R}$ is continuous and satisfies $\int_0^1 x^n g(x)dx = 0$ for all $n \geq 1$. Prove that $g(x) = 0$ for all $x \in [0, 1]$.

3. (10 points) For a sequence $\{x_n, n \in \mathbb{N}\}$ of real numbers, let S be the set of *subsequential* limits of $\{x_n\}$.
(a) Prove there is a sequence $\{x_n\}$ for which $S = [0, 1]$.
(b) Prove that for any sequence $\{x_n\}$, the set S is closed.

4. (10 points) If $f, g : [-\pi, \pi] \to \mathbb{C}$ are continuous, denote $\langle f, g\rangle = \int_{\pi}^{\pi} f(x)\overline{g(x)}\, dx$, and recall that the *Fourier coefficients* of f are defined by $\hat{f}(m) = \int_{-\pi}^{\pi} f(x)\frac{e^{-imx}}{\sqrt{2\pi}}\, dx$, for $m \in \mathbb{Z}$. Let $\{f_n\}$ be a sequence of $\mathbb{C}$-valued continuous functions such that $\int_{-\pi}^{\pi} |f_n(x)|^2\, dx \leq 1$ for all $n \in \mathbb{N}$.
(a) Show that there is a subsequence $\{f_{n_k}\}$ such that for each $m \in \mathbb{Z}$, $\{\hat{f}_{n_k}(m) : k \in \mathbb{N}\}$ is a convergent sequence of complex numbers.
(b) Show that for a subsequence as in (a) one in fact has convergence of the complex-valued sequence $\{\langle f_{n_k}, g\rangle\}$ as $k \to \infty$ for every continuous $g : [-\pi, \pi] \to \mathbb{C}$.

II. Complex Analysis

Do all 3 questions

5. (10 points) Let $f(z) = \frac{1}{1+z^5}$.
(a) If Γ_R is the straight line segment in the complex plane from 0 to $Re^{2\pi i/5}$, prove that

$$\int_{\Gamma_R} f(z)\,dz = e^{2\pi i/5} \int_0^R f(x)dx.$$

(b) Evaluate $\int_0^\infty f(x)\,dx$.

6. (10 points) (a) Prove that if f is a non-constant entire function, then it's image is dense in $\mathbb{C}$.
(b) Let g be an entire function so that $g(x) = g(x+1)$ for every real x. Is it necessarily the case that $g(z) = g(z+1)$ for every $z \in \mathbb{C}$? Prove or give a counter-example.

7. (10 points) Suppose D is a bounded open connected subset of $\mathbb{C}$ and f is a continuous $\mathbb{C}$-valued function on $D \cup \partial D$ which is analytic on D. Suppose for every $z \in \partial D$ we have $|f(z)| \leq 1$. Let $\rho(z)$ be the distance from z to ∂D.
(a) Prove that $|f'(z)| \leq 1/\rho(z)$ for all $z \in D$.
(b) Does the same always hold if D is the upper half plane?