The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
January 6, 2018

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1. (10 points) Find an $n \times n$-matrix P with real entries, such that $P^T = P$, $P^2 = P$, and whose null space is spanned by the vector $(1, \dots, 1)^T$.

2. (10 points) Let A be an $n \times n$ matrix with complex entries. Suppose that m is a positive integer such that A^m is diagonalizable. Prove that A^{m+1} is diagonalizable.

3. (10 points) Suppose that A is a 2×2 matrix with real entries. Suppose that $\det A = 1$, and that A does not have a real eigenvalue. Prove that there exists an invertible 2×2-matrix S, with real entries, such that $S^{-1}AS$ is equal to the rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, for some $\theta \in (0, \pi) \subset \mathbb{R}$.

4. (10 points) Let
$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}, \quad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$
For which values of $\vec{x}(0)$ does the equation $\vec{x}\,'(t) = A\vec{x}(t)$ have $\vec{x}(t) \to \vec{0}$ as $t \to \infty$?

5. (10 points) Consider a real valued function $x = x(t)$ for $t \geq 0$ that satisfies the ODE $x'(t) = (x(t))^{1/2}$ and the intial condition $x(0) = 0$.

   (i) Show that $x(t) = 0$ for all t satisfies the above ODE and initial condition.

   (ii) Find a nonzero solution $x(t)$ to the same ODE and initial condition above.

   (iii) Show that there are infinitely many solutions $x(t)$ to the same ODE and initial condition above.

   (iv) The ODE $x' = g(x, t)$ subject to $x(t_0) = x_0$ for given real x_0, t_0 is known to have a unique solution for t near t_0 for sufficiently well-behaved $g(x, t)$. What is the usual sufficient condition, and why doesn't the above example violate this principle?

6. (10 points) Let $f(x)$ be a differentiable function with $f(0) = f(1) = 0$. Consider the wave equation
$$u_{tt}(x, t) = u_{xx}(x, t)$$
for $0 \leq x \leq 1$ and $t \geq 0$ subject to the (Dirichlet) boundary conditions $u(0, t) = u(1, t) = 0$ and
$$u(x, 0) = f(x), \quad u_t(x, 0) = 0$$
for all $0 \leq x \leq 1$.

   (i) Use separation of variables to solve this PDE.

   (ii) Which of the terms in the above solution are symmetric about $x = 1/2$, i.e., which of the solutions $F(x)G(t)$ to the above wave equation have $F(x) = F(1 - x)$? Which satisfy $F(x) = -F(1 - x)$?

   (iii) Say that $f(x) = f(1 - x)$. Which terms in the summation in the general separation of variables must be zero, and which are not necessarily zero?