The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
September 5, 2017

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1. (10 points) Consider the equation $y'' + 2y^3 + 2y = 0$, where $'$ denotes d/dt.

   (i) Find a function $g(y)$ such that the quantity $f(t) = g(y(t)) + (y'(t))^2$ remains invariant.

   (ii) Show that for any initial conditions $y(0), y'(0)$, the solution $y(t)$ is bounded for all $t > 0$.

2. (10 points) Consider the equation $y' = g(t, y)$ where $g(t, y) = 2y - \sin(2^t y^{2017})$ and where $'$ denotes d/dt, under the initial condition $y(0) = 1$. Assuming that there exists a solution $y(t)$ for all $t \geq 0$ (differentiable in t), prove that $y(t) \to \infty$ as $t \to \infty$. [Hint: this is not hard if you can prove that $y(t) \geq 1$ for all $t \geq 0$.]

3. (10 points) Consider the heat equation

   $$u_t = u_{xx}$$

   for $t \geq 0$ and $-1 \leq x \leq 1$, and boundary conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = f(x)$ where $f(x)$ is an arbitrary (real valued) differentiable function in $[0, 1]$.

   (i) Use separation of variables to find a general solution to this PDE.

   (ii) Show that if f is continuous on $[0, 1]$, then the solution obtained by separation of variables is a series that converges pointwise at every point (x, t) with $t > 0$.

4. (10 points) Let $A \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix (i.e., $A^T = A^{-1}$). Suppose that $\frac{1+i}{\sqrt{2}}$ is one eigenvalue of A.

   (i) What are eigenvalues of A^2?

   (ii) Is there necessarily a solution to the following equation?

   $$A^{38} = \alpha A^4 + \beta A^2 + \gamma I.$$

   If so, find $\alpha, \beta, \gamma \in \mathbb{R}$ that solve the equation.
   Hint: The Cayley-Hamilton theorem will help determine the answer. We state this theorem below.

   Theorem 1 (Cayley-Hamilton Theorem) *Given a square matrix $A \in \mathbb{R}^{n\times n}$, let $p(\lambda) := \det(\lambda I - A)$ be the characteristic polynomial. Then A satisfies the characteristic equation. To be precise, write $p(\lambda)$ as a sum of monomials*

   $$p(\lambda) = \sum_{i=0}^{n} \alpha_i \lambda^i.$$

   Then

   $$p(A) := \sum_{i=0}^{n} \alpha_i A^i = 0.$$

5. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}.$$

(i) What is $\operatorname{rank}(A)$?

(ii) Find an orthonormal basis of eigenvectors for A.

(iii) Let $B = \exp(A)$. What is the value of $B_{1,1}$ (the upper left entry of B)?

6. (10 points) Define the map $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by the equation given in each of the following subproblems. Determine whether the map is a linear transformation. If not, show that it is not. If so, determine the dimension of the image (also called the range) and the dimension of the kernel (also called the null space). Below $A \in \mathbb{R}^{n\times n}$.

(i) $f(A) = A + A^{-1}$.

(ii) $f(A) = Avv^T$ where $v \in \mathbb{R}^n$ and $v \neq 0$. Also, v^T is the transpose of v.

(iii) $f(A) = D{\cdot}A - D^{-1}{\cdot}A^T$, where D is a diagonal matrix satisfying $D_{i,i} = (-1)^i$, $i = 1, 2, \ldots, n$.