The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
September 5, 2017

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1. (10 points) Let

$$\vec{F} = (-yz\sin 2x - x^2 y)\,\hat{\boldsymbol{\imath}} + (z\cos^2 x)\,\hat{\boldsymbol{\jmath}} + (e^{z^2} + y\cos^2 x)\,\hat{\mathbf{k}}\ .$$

Let C be the curve given by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x+2y+z = 6$, oriented so that its projection on the plane $z = 0$ is oriented anticlockwise. Calculate

$$\oint_C \mathbf{F}.\mathbf{dr}.$$

2. (10 points) For each of the following questions, if the answer is yes, give a proof; if the answer is no, give a counterexample with proof.

In each of the following questions, f_n is a sequence of continuous functions on $[0, 1]$.

(i) Assume that f_n converges uniformly to a function f. Must f be continuous?

(ii) Assume that f_n converges pointwise to a continuous function f. Must f_n converge uniformly to f?

(iii) Assume that f_n is a monotone decreasing sequence (i.e., for all $x \in [0, 1]$ and all n, $f_n(x) \geq f_{n+1}(x)$), that converges pointwise to a function f. Must f be continuous?

3. (10 points) Say that a non-negative sequence (b_n) is *super-summable* if for every non-negative sequence (a_n) such that $a_n \to 0$ we have $\sum_n a_n b_n$ converges. Prove that (b_n) is super-summable if and only if $\sum b_n$ converges.

4. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\ .$$

5. **Instruction.** Carefully specify any branches or branch cuts you may use.

(A) Let $H = \{z \in \mathbb{C} \colon \mathrm{Re}(z) \geq 0\} \setminus \{0\}$. Write down, explicitly in terms of elementary functions, a continuous function $h : H \to \mathbb{R}$, with the properties:

(i) h is harmonic in the interior of H,

(ii) $h(z) \equiv 1$ on the positive imaginary axis,

(iii) $h(z) \equiv 0$ on the negative imaginary axis.

(B) Let $D = \{z \in \mathbb{C}\colon |z| \leq 1\} \setminus \{1, -1\}$. Write down, explicitly in terms of elementary functions, a continuous function $f : D \to \mathbb{R}$, with the properties:

(i) f is harmonic in the interior of D,

(ii) $f(z) = 1$, for $|z| = 1$ and $\mathrm{Im}(z) > 0$,

(iii) $f(z) = -1$, for $|z| = 1$ and $\mathrm{Im}(z) < 0$.

6. Let $D = \{z \in \mathbb{C}\colon |z| < 1\}$.

(i) Write down a holomorphic (i.e., complex analytic) function $f(z) : D \to D$ with the property that $f(-\frac{1}{2}) = 0$ and $f(0) = \frac{1}{2}$.

(ii) Let $g : D \to D$ be an arbitrary holomorphic (i.e., complex analytic) function such that $g(-\frac{1}{2}) = 0$ and $g(0) = \frac{1}{2}$. Find $g(\frac{1}{2})$.